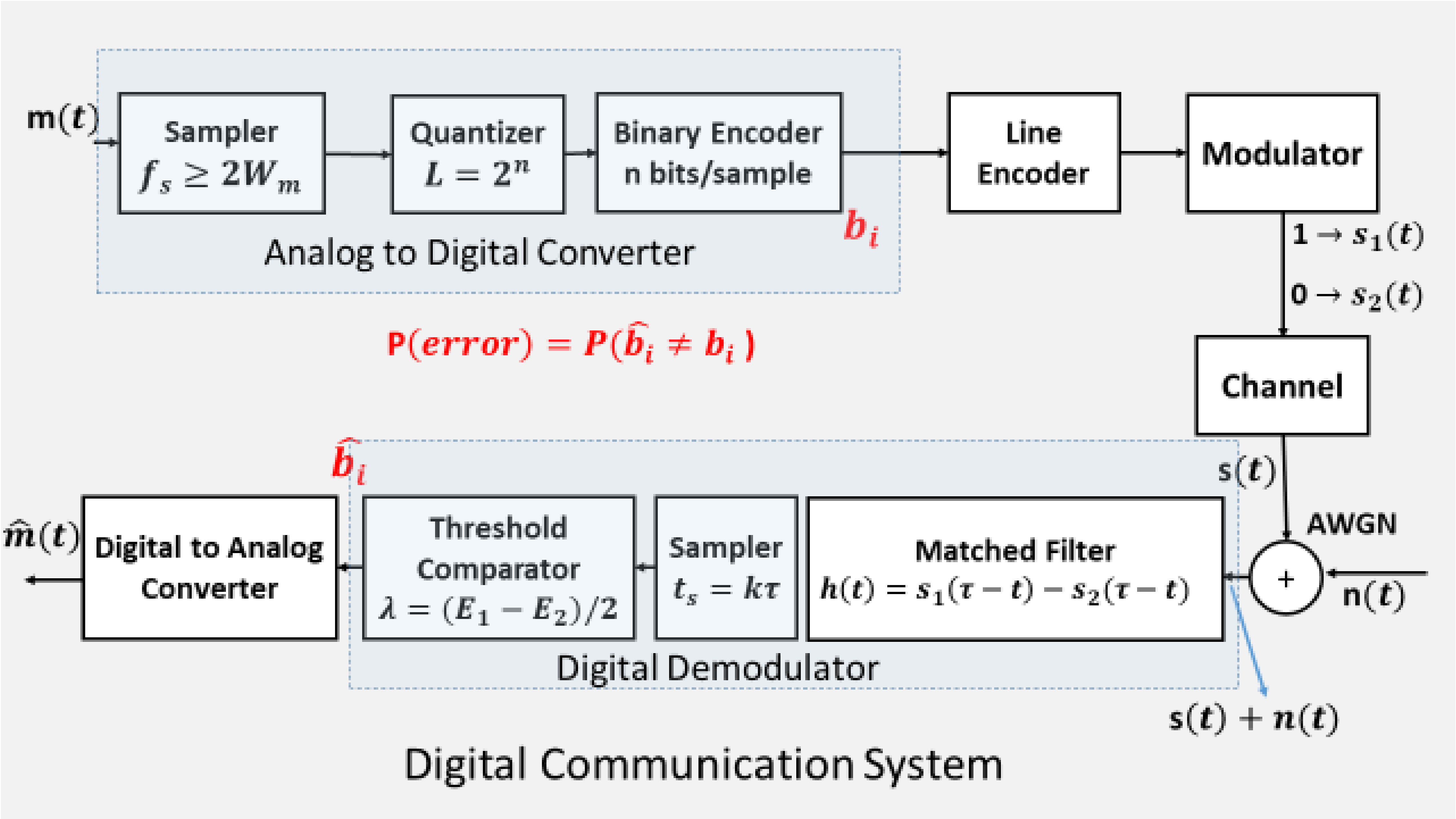
M-Ary Data Transmission

Topics to be covered in this video and subsequent ones

- Orthogonal Functions and Signal Space Representation
- Gram-Schmidt Orthogonaliztion Procedure
- Optimum Receiver for Binary Transmission (Revisited using signal space concept)
- Optimum Receiver for M-Ary Transmission (using signal space representation)
- M-ary Coherent Amplitude-Shift Keying (M-ASK)
- M-ary Coherent Phase-Shift Keying (M-PSK)
- M-ary Coherent Frequency-Shift Keying (M-FSK)
- M-ary Quadrature Amplitude Modulation (M-QAM)
- Union Bound on the Symbol Probability of Error
- Comparison of the various M-ary modulation techniques

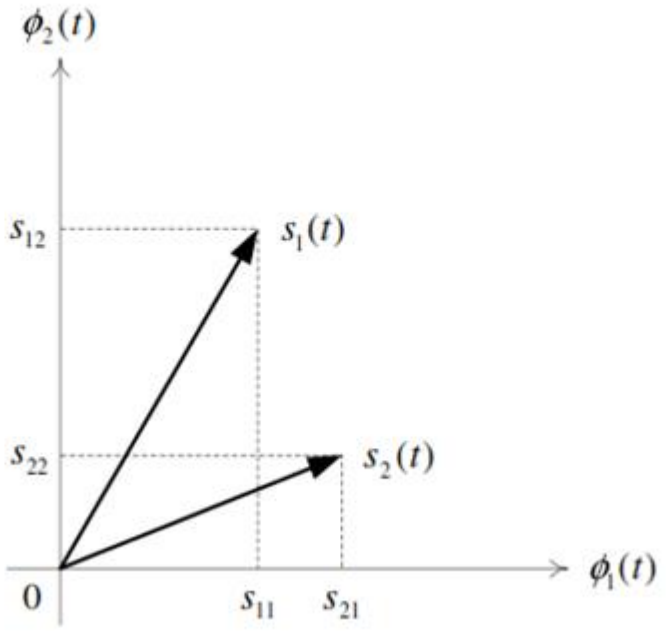
The Binary Communication System (Revisited)



Assumptions

- In binary data transmission over a communication channel, logic 1 is represented by a signal $s_1(t)$ and logic 0 by a signal $s_2(t)$.
- The time allocated for each signal is the bit duration T_b(τ in the previous chapter) in the case of binary and T_s for the case of M-ary.
- The data rate is $R_b = 1/T_b$ bits/sec.
- The channel noise $n(t)$ is additive white Gaussian (AWGN) with a double-sided PSD of $N_0/2$ W/Hz, mean $\mathrm{E}\{n(t)\} = 0$, $R_n(\tau) = \frac{N_0}{2}\delta(\tau)$. Noise is assumed to be added at the front end of the receiver (with variance $N_0/2$).
- The data component at the front end of the receiver is assumed to be an exact replica of the transmitted signal, in the sense that the transmission bandwidth of the medium is wide enough to reproduce the signal without distortion.
- Bits in different time intervals are assumed independent.
- The signal to be processed by the receiver is the noisy signal $\boldsymbol{y(t) = s_i(t) + n(t)}$
- **Based on $y(t)$, the task of the receiver is to decide whether a 1 or a 0 was transmitted during each transmission slot $\boldsymbol{\tau}$ with minimum probability of error.**
- The approach is based on the signal space where time functions are represented by numbers, which may be deterministic or random depending on the type of signal.

Geometric Representation of Signals (Signal Space Concept)



Wish to represent two arbitrary signals $s_1(t)$ and $s_2(t)$ as *linear combinations* of two *orthonormal* basis functions $\phi_1(t)$ and $\phi_2(t)$.

- $\phi_1(t)$ and $\phi_2(t)$ are orthonormal if:

$$\int_0^{T_b} \phi_1(t)\phi_2(t)\mathrm{d}t = 0 \ (\textit{orthogonality}),$$

$$\int_0^{T_b} \phi_1^2(t)\mathrm{d}t = \int_0^{T_b} \phi_2^2(t)\mathrm{d}t = 1 \ (\textit{normalized} \text{ to have unit energy}).$$

$$< \emptyset_1(t), \emptyset_1(t) > = 1$$
$$< \emptyset_2(t), \emptyset_2(t) > = 1$$
$$< \emptyset_1(t), \emptyset_2(t) > = 0$$

- The representations are

Signal components. These are numbers

$$s_1(t) = s_{11}\phi_1(t) + s_{12}\phi_2(t),$$
$$s_2(t) = s_{21}\phi_1(t) + s_{22}\phi_2(t).$$

where $s_{ij} = \int_0^{T_b} s_i(t)\phi_j(t)\mathrm{d}t, \quad i,j \in \{1,2\},$

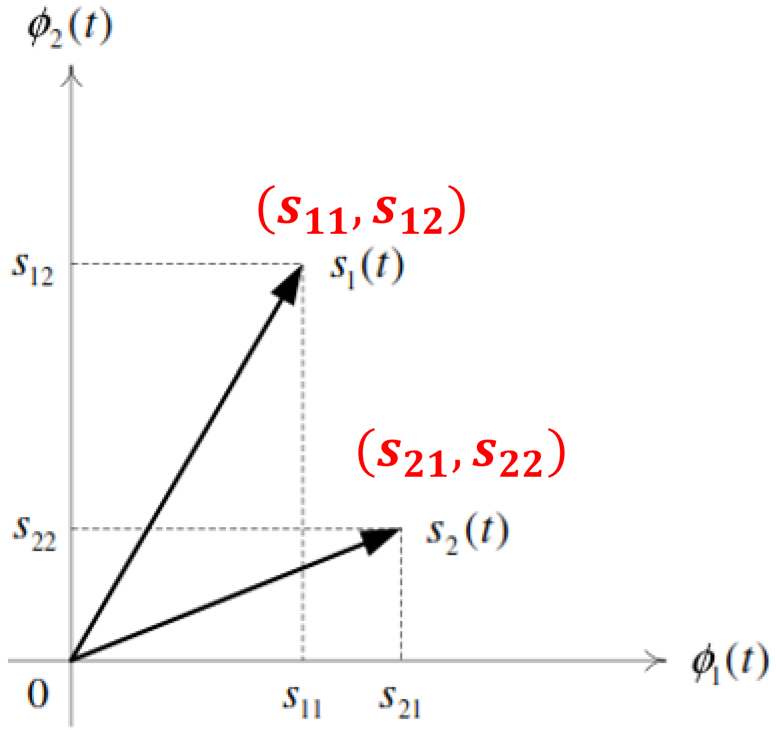
To find s_{11} multiply both sides by $\emptyset_1(t)$, integrate over (0, T_b), and make use of the properties of the basis functions.

$$\int_0^{T_b} s_1(t)\emptyset_1(t)dt = \overbrace{\int_0^{T_b} s_{11}\emptyset_1(t)\emptyset_1(t)dt}^{= s_{11}} + \overbrace{\int_0^{T_b} s_{12}\emptyset_2(t)\emptyset_1(t)dt}^{= 0}$$

Signal Space Representation: Energy, Distance, and Probability of Error

- $E_1 = \int_0^{T_b} (s_1(t))^2 dt = \int_0^{T_b} \left(s_{11}\emptyset_1(t) + s_{12}\emptyset_2(t)\right)^2 dt,$
- $E_1 = \int_0^{T_b} (s_1(t))^2 dt = (s_{11})^2 + (s_{12})^2,$ **Signal Energy**
- $E_2 = \int_0^{T_b} (s_2(t))^2 dt = \int_0^{T_b} \left(s_{21}\emptyset_1(t) + s_{22}\emptyset_2(t)\right)^2 dt$
- $E_2 = \int_0^{T_b} (s_2(t))^2 dt = (s_{21})^2 + (s_{22})^2,$ **Signal Energy**
- $d_{12}^2 = \int_0^{T_b} (s_1(t) - s_2(t))^2 dt = (s_{11} - s_{21})^2 + (s_{12} - s_{22})^2,$ **Square of the Distance Between two Signals**
- $P_b^* = Q\left(\sqrt{\frac{\int_0^{T_b} \left(s_1(t) - s_2(t)\right)^2 dt}{2N_0}}\right) = Q\left(\frac{d_{12}}{\sqrt{2N_0}}\right)$; **Bit Error Probability**

Geometric Representation of Signals: Summary



$$s_1(t) = s_{11}\phi_1(t) + s_{12}\phi_2(t),$$

Signal Representation

$$s_2(t) = s_{21}\phi_1(t) + s_{22}\phi_2(t),$$

$$s_{ij} = \int_0^{T_b} s_i(t)\phi_j(t)\mathrm{d}t,\ i,j \in \{1,2\},$$

Signal Coefficients

$$d_{12}^2 = \int_0^{T_b} (s_1(t) - s_2(t))^2 dt$$
$$= (s_{11} - s_{21})^2 + (s_{12} - s_{22})^2$$

$$E_1 = \int_0^{T_b} (s_1(t)^2 dt = (s_{11})^2 + (s_{12})^2$$

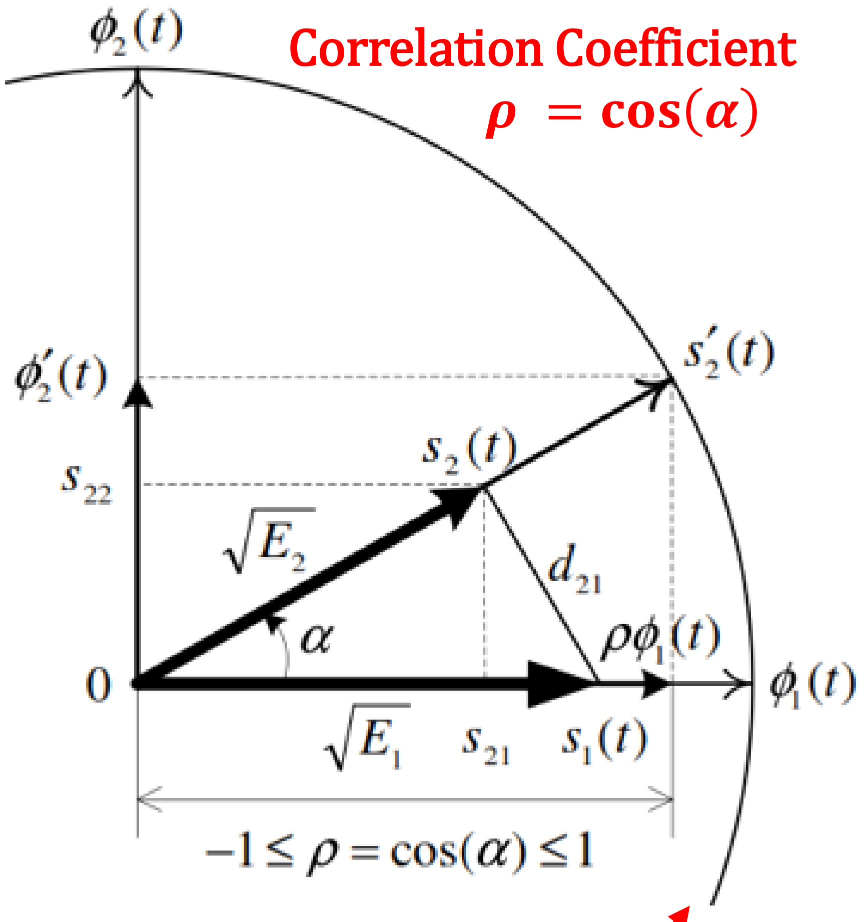
$$E_2 = \int_0^{T_b} (s_2(t)^2 dt = (s_{21})^2 + (s_{22})^2$$

$$P_b^* = Q\left(\sqrt{\frac{\int_0^{T_b} (s_1(t) - s_2(t))^2 dt}{2N_0}}\right) = Q\left(\frac{d_{12}}{\sqrt{2N_0}}\right)$$

Gram-Schmidt Method: Basis for a Two-Dimensional Space

- The Gram-Schmidt method is a procedure for generating a set of orthonormal basis functions from a given set of functions ($s_1(t), s_2(t)$)
- The original set of functions may be dependent or independent, but the basis functions are both
- Linearly independent and
- Orthonormal.

Gram-Schmidt Method: Basis for a Two-Dimensional Space



Correlation Coefficient

$\boldsymbol{\rho = \cos(\alpha)}$

Unit Circle

$s_2' = \textit{component} \parallel \textit{to } \phi_1(t)$
$+\textit{component} \perp \textit{to } \phi_1(t)$

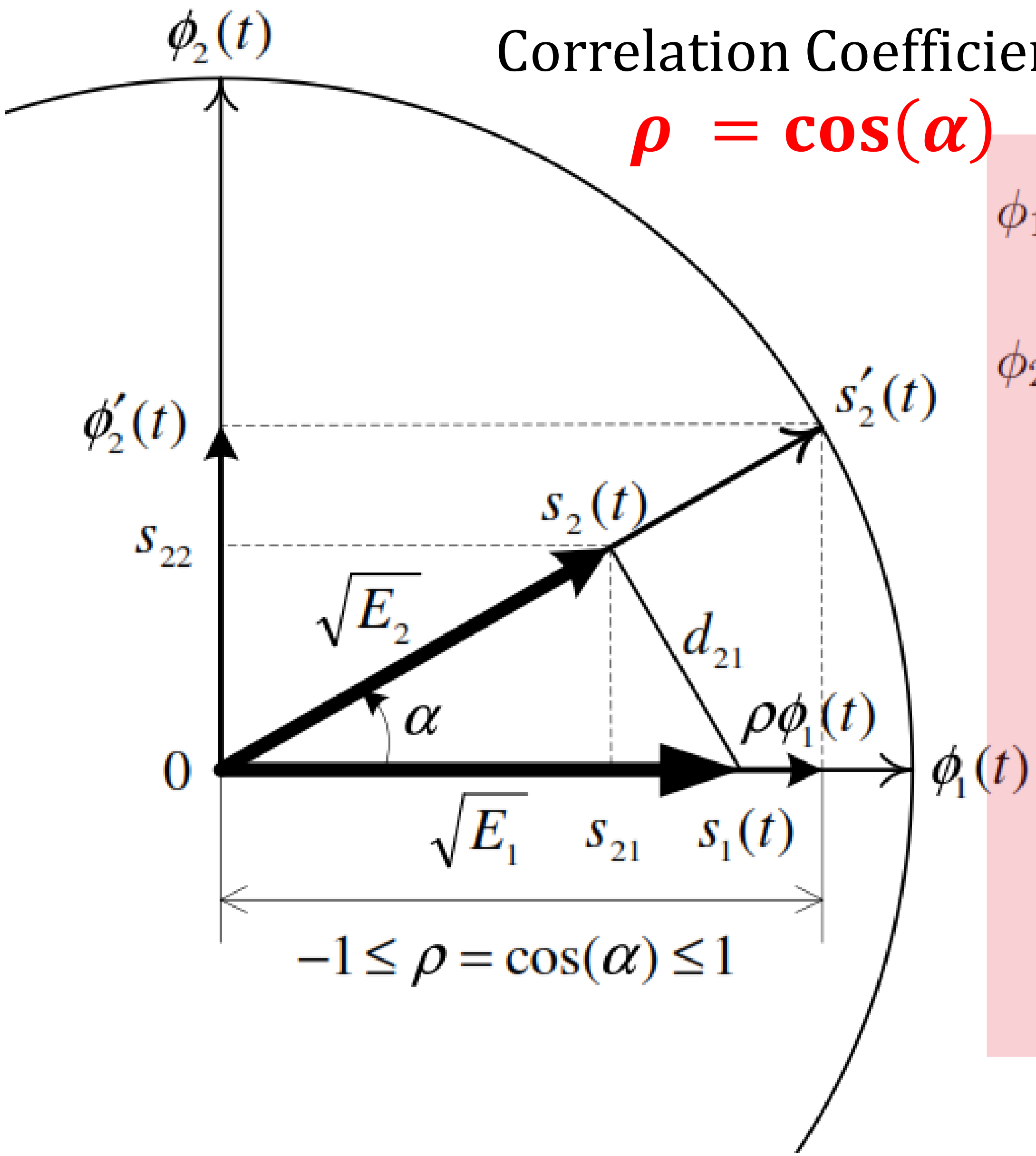
1. Let $\phi_1(t) \equiv \dfrac{s_1(t)}{\sqrt{E_1}}$. Note that $s_{11} = \sqrt{E_1}$ and $s_{12} = 0$.
2. Project $s_2'(t) = \dfrac{s_2(t)}{\sqrt{E_2}}$ onto $\phi_1(t)$ to obtain the *correlation coefficient*:

$$\rho = \int_0^{T_b} \frac{s_2(t)}{\sqrt{E_2}} \phi_1(t) \mathrm{d}t = \frac{1}{\sqrt{E_1 E_2}} \int_0^{T_b} s_1(t) s_2(t) \mathrm{d}t.$$

3. Subtract $\rho\phi_1(t)$ from $s_2'(t)$ to obtain $\phi_2'(t) = \frac{s_2(t)}{\sqrt{E_2}} - \rho\phi_1(t)$.
4. Finally, normalize $\phi_2'(t)$ to obtain:

$$\begin{aligned} \phi_2(t) &= \frac{\phi_2'(t)}{\sqrt{\int_0^{T_b} \left[\phi_2'(t)\right]^2 \mathrm{d}t}} = \frac{\phi_2'(t)}{\sqrt{1 - \rho^2}} \\ &= \frac{1}{\sqrt{1-\rho^2}} \left[\frac{s_2(t)}{\sqrt{E_2}} - \frac{\rho s_1(t)}{\sqrt{E_1}} \right]. \end{aligned}$$

Gram-Schmidt Method: Basis for a Two-Dimensional Space



Correlation Coefficient

$$\rho = \cos(\alpha)$$

$$\rho = \frac{1}{\sqrt{E_1 E_2}} \int_0^{T_b} s_1(t) s_2(t)\, dt$$

$$\phi_1(t) = \frac{s_1(t)}{\sqrt{E_1}}, \qquad s_{11} = \sqrt{E_1} \text{ and } s_{12} = 0.$$

$$\phi_2(t) = \frac{1}{\sqrt{1-\rho^2}} \left[\frac{s_2(t)}{\sqrt{E_2}} - \frac{\rho s_1(t)}{\sqrt{E_1}} \right],$$

$$s_{21} = \int_0^{T_b} s_2(t)\phi_1(t)\mathrm{d}t = \rho\sqrt{E_2}, \quad \sqrt{E_2}\cos(\alpha)$$

$$s_{22} = \left(\sqrt{1-\rho^2}\right)\sqrt{E_2}, \qquad \sqrt{E_2}\sin(\alpha)$$

$$d_{21} = \sqrt{\int_0^{T_b} [s_2(t) - s_1(t)]^2 \mathrm{d}t}$$

$$= E_1 - 2\rho\sqrt{E_1 E_2} + E_2.$$

Gram-Schmidt Method: M-Ary Case

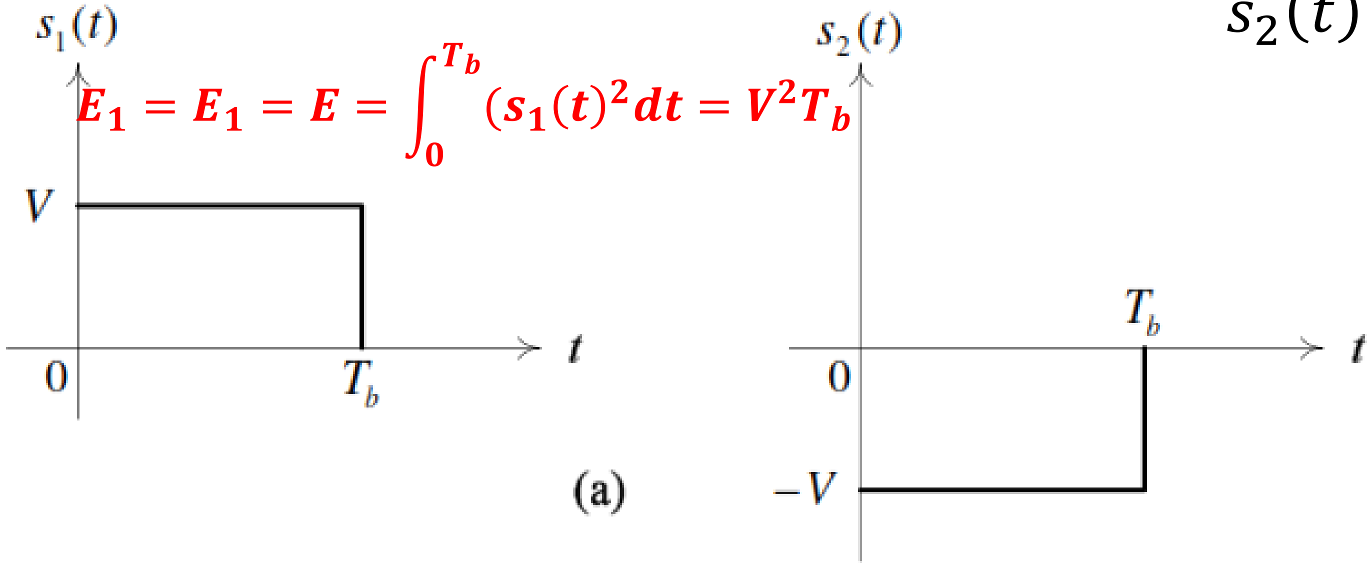
$$
\begin{aligned}
\phi_1(t) &= \frac{s_1(t)}{\sqrt{\int_{-\infty}^{\infty} s_1^2(t)\mathrm{d}t}}, \\
\phi_i(t) &= \frac{\phi_i'(t)}{\sqrt{\int_{-\infty}^{\infty} \left[\phi_i'(t)\right]^2 \mathrm{d}t}}, \quad i = 2, 3, \ldots, N, \\
\phi_i'(t) &= \frac{s_i(t)}{\sqrt{E_i}} - \sum_{j=1}^{i-1} \rho_{ij}\phi_j(t), \\
\rho_{ij} &= \int_{-\infty}^{\infty} \frac{s_i(t)}{\sqrt{E_i}}\phi_j(t)\mathrm{d}t, \quad j = 1, 2, \ldots, i-1.
\end{aligned}
$$

If the waveforms $\{s_i(t)\}_{i=1}^{M}$ form a *linearly independent set*, then $N = M$. Otherwise $N < M$.

Example: Polar Non-return to zero Binary Signals

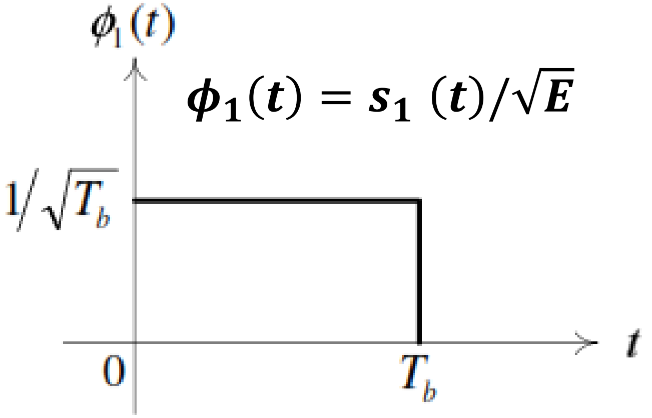
$s_2(t) = -s_1(t)$; **Linearly Dependent**

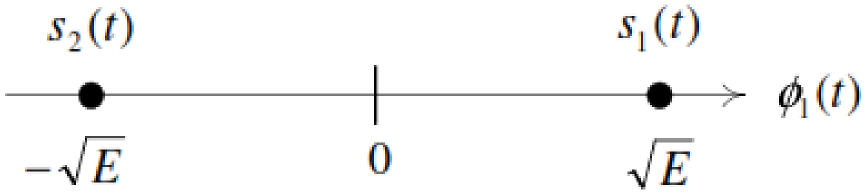
$$E_1 = E_1 = E = \int_0^{T_b} (s_1(t))^2 dt = V^2 T_b$$



$$\rho = \frac{1}{\sqrt{E_1 E_2}} \int_0^{T_b} s_1(t) s_2(t) dt = -1$$

$$\phi_1(t) = s_1(t)/\sqrt{E}$$





s_{21} s_{11}

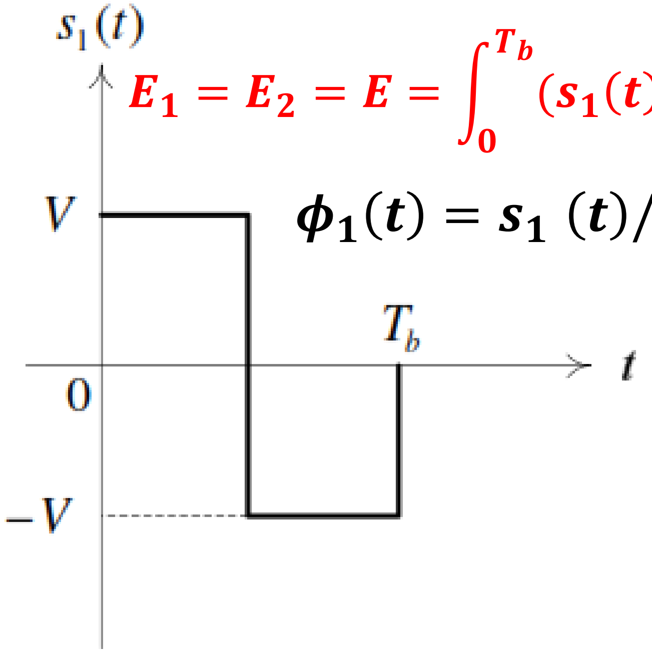
For the case of binary antipodal signaling, i.e., when $s_2(t) = -s_1(t)$, we need only one basis function since signals are linearly dependent. The signals are represented as:

$$s_1(t) = \sqrt{E}\phi_1(t)$$
$$s_2(t) = -\sqrt{E}\phi_1(t)$$

(a) Signal set. (b) Orthonormal function. (c) Signal space representation.

Example: Orthogonal Binary Signals

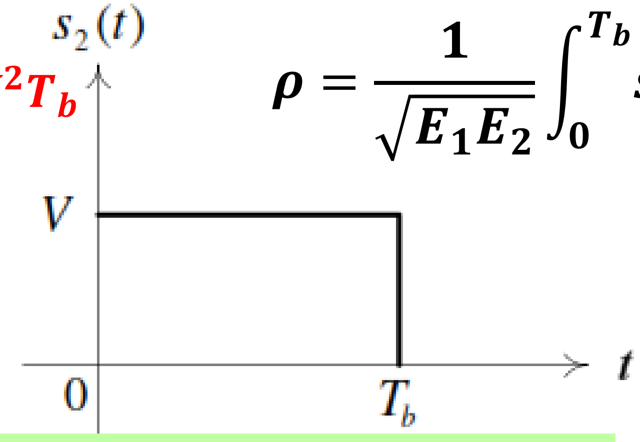
$$E_1 = E_2 = E = \int_0^{T_b} (s_1(t))^2 dt = V^2 T_b$$



$$\phi_1(t) = s_1\ (t)/\sqrt{E}$$

$$\rho = \frac{1}{\sqrt{E_1 E_2}} \int_0^{T_b} s_1(t) s_2(t) dt = 0$$

Linearly independent and orthogonal



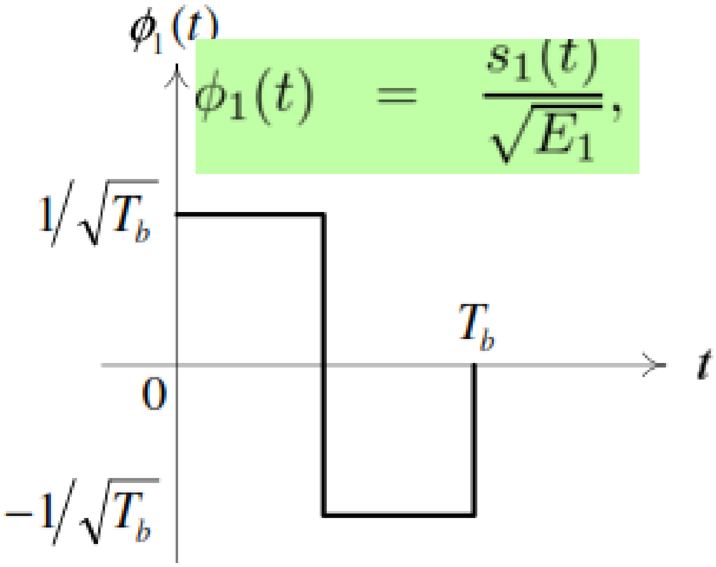
$$\phi_2(t) \quad = \quad \frac{1}{\sqrt{1-\rho^2}}\left[\frac{s_2(t)}{\sqrt{E_2}} - \frac{\rho s_1(t)}{\sqrt{E_1}}\right],$$

(a)

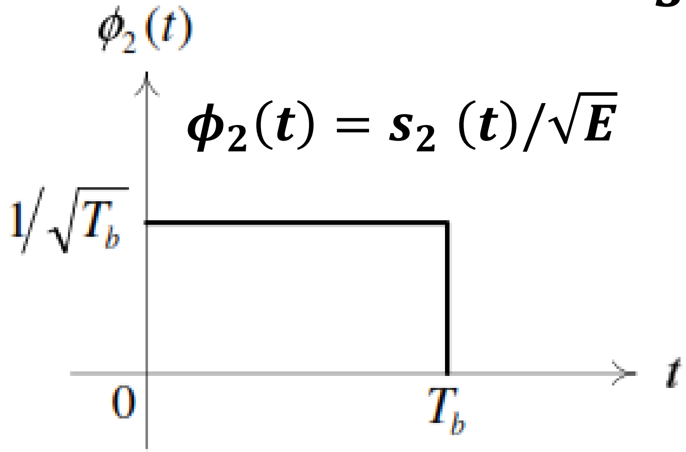
For the case of binary signaling when ρ =0 we need two bases functions. The signals are represented as:

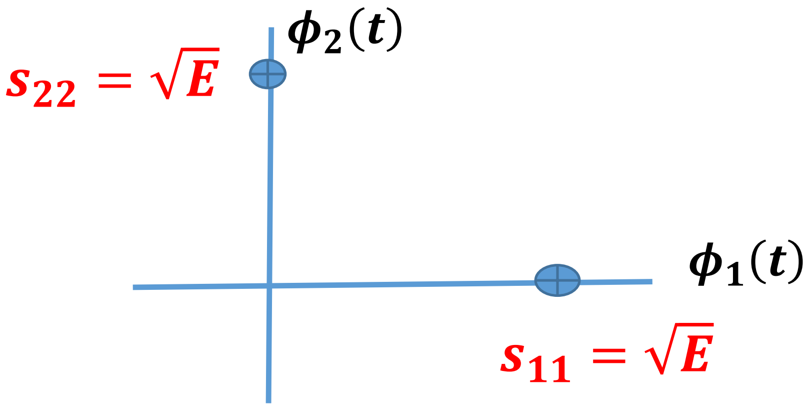
$$s_1(t) = \sqrt{E}\phi_1(t)$$
$$s_2(t) = \sqrt{E}\phi_2(t)$$

$$\phi_1(t) \quad = \quad \frac{s_1(t)}{\sqrt{E_1}},$$



$$\phi_2(t) = s_2\ (t)/\sqrt{E}$$



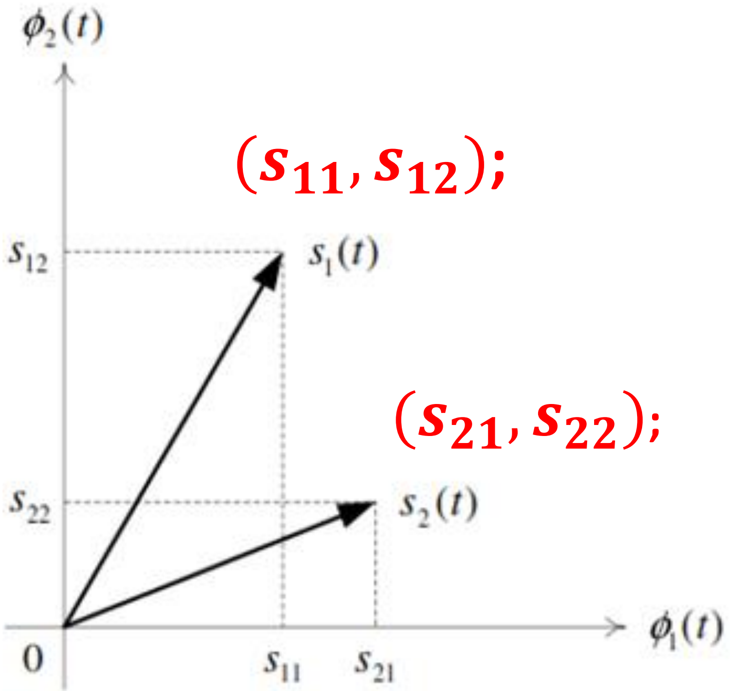


Signal Space Representation of Signals: Revisited

- In the previous video, we saw that if a signal space is characterized by two basis functions $\emptyset_1(t)$ and $\emptyset_2(t)$, and if we are given two signals $s_1(t)$ and $s_2(t)$, then $s_1(t)$ and $s_2(t)$ can be expressed as:

$$\begin{aligned} s_1(t) &= s_{11}\phi_1(t) + s_{12}\phi_2(t), \\ s_2(t) &= s_{21}\phi_1(t) + s_{22}\phi_2(t). \end{aligned} \qquad (1)$$

where $$s_{ij} = \int_0^{T_b} s_i(t)\phi_j(t)\mathrm{d}t, \quad i,j \in \{1,2\}, \qquad (2)$$



- If the receiver is able to recover the coefficients s_{ij} using (2), then according to (1) the signals are completely known. This is of course true in the absence of noise. However, in the presence of noise, a random variable is added to s_{ij} that makes the decision more involved.
- The objective of the receiver is to retrieve the coefficients s_{ij} using (2) and to make a decision accordingly

Receiver Structure

- Since each signal is characterized by two coefficients in the $\emptyset_1(t)$ and $\emptyset_2(t)$ plane, then we need two correlators that retrieve these coefficients at the receiver (figure below)
- However, in the presence of noise, the received signal is the sum of the transmitted signal and the AWGN component. That is: r(t) $= s_i(t) + w(t)$.
- The components in the $\emptyset_1(t)$ and $\emptyset_2(t)$ directions are:

$$r_1 = \int_0^{T_b} (s_i(t) + w(t))\emptyset_1(t)dt \; = s_{i1} + N_1; \; N_1 \sim N(0, N_0|2); \; r_1 \sim N(s_{i1}, N_0|2);$$

$$r_2 = \int_0^{T_b} (s_i(t) + w(t))\emptyset_2(t)dt \; = s_{i2} + N_2; \; N_2 \sim N(0, N_0|2); \; r_2 \sim N(s_{i2}, N_0|2);$$

r_1 *and* r_2 are independent

If s_1 is sent, then

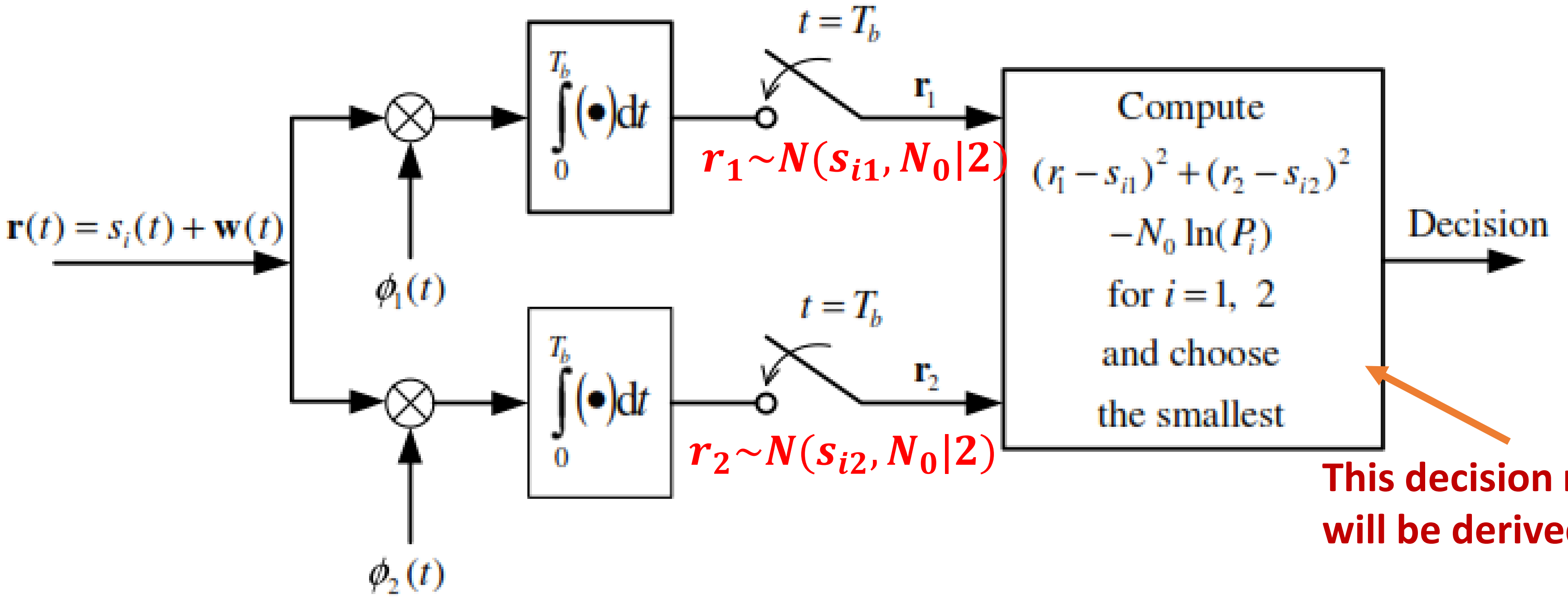
$r_1 \sim N(s_{11}, N_0|2);$

$r_2 \sim N(s_{12}, N_0|2);$

If s_2 is sent, then

$r_1 \sim N(s_{21}, N_0|2);$

$r_2 \sim N(s_{22}, N_0|2);$



This decision rule will be derived later.

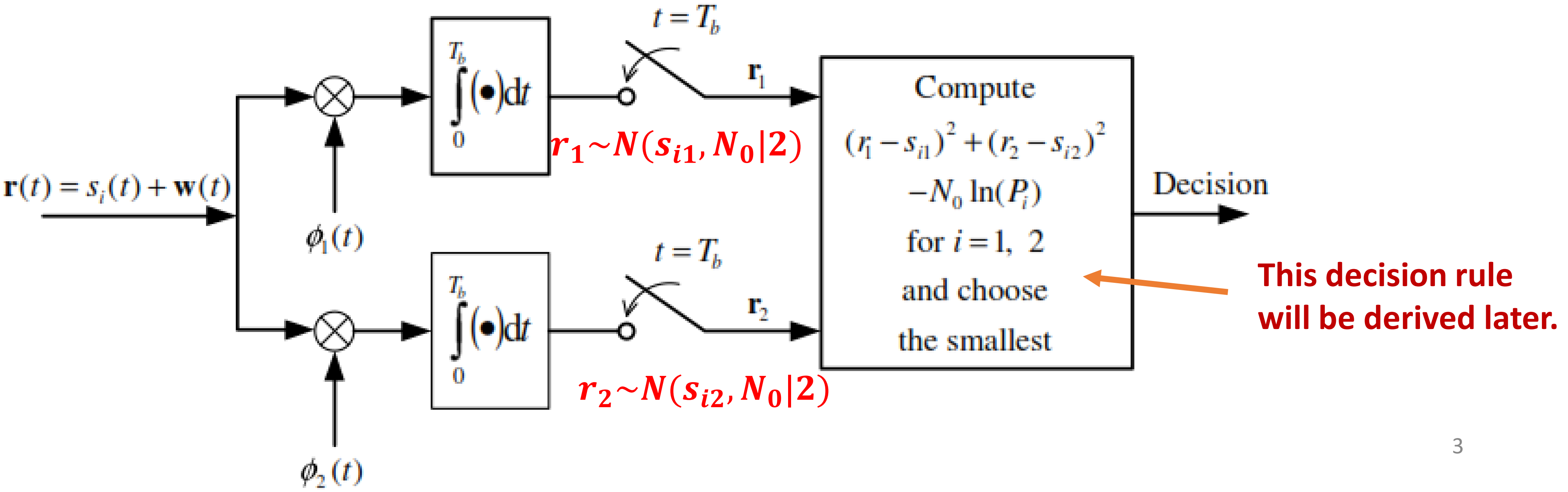
The Minimum Probability of Error Receiver

- Given the correlator outputs r_1 and r_2, we need to find the decision rule that minimizes the probability of error

$$r_1 = \int_0^{T_b} (s_i(t) + w(t)) \emptyset_1(t) dt = s_{i1} + N_1; \; N_1 \sim N(0, N_0|2); \; r_1 \sim N(s_{i1}, N_0|2);$$
$$r_2 = \int_0^{T_b} (s_i(t) + w(t)) \emptyset_2(t) dt = s_{i2} + N_2; \; N_2 \sim N(0, N_0|2); \; r_2 \sim N(s_{i2}, N_0|2);$$

r_1 *and* r_2 are independent

- The decision rule will be function of the received observations r_1 and r_2, the signal components s_{ij}, and the noise power N_0 as we shall derive next.

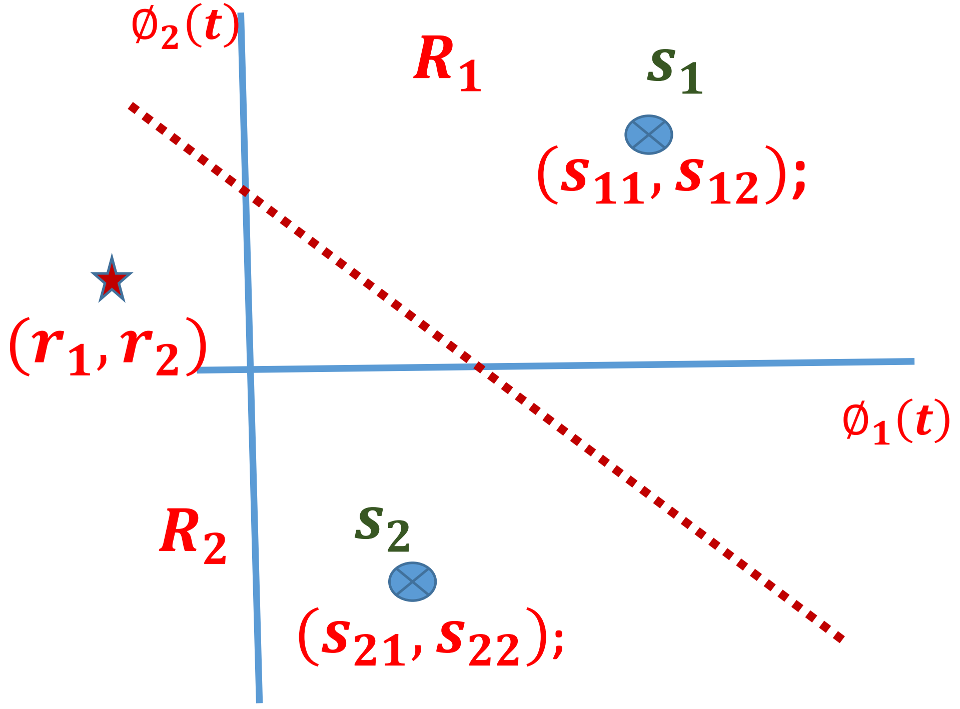


The Optimum Decision Rule

- Let P_1 and P_2 be the probability of sending signals s_1 and s_2, respectively.
- Let R_1 and R_2 be the decision regions in the two-dimensional space corresponding to signals s_1 and s_2, respectively.
- If $(r_1, r_2 \in R_1)$ decide s_1 (digit 1) Also. Else, decide s_2 (digit 0).

$$P_b = P(send\ s_1, decide\ s_2) + P(send\ s_2, decide\ s_1)$$

$$P_b = P_1 P(decide\ s_2|s_1) + P_2 P(decide\ s_1|s_2)$$



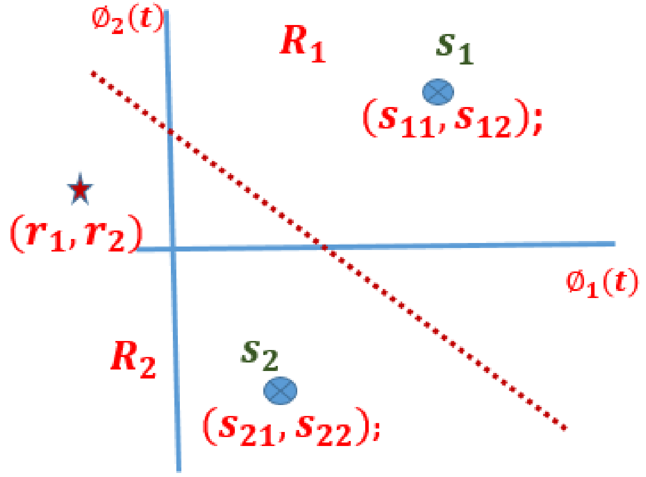
The Optimum Decision Rule

$$P_b = P_1 \int_{R_2} f(r_1, r_2 | b_i = 1) dr_1 dr_2 + P_2 \int_{R_1} f(r_1, r_2 | b_i = 0) dr_1 dr_2$$

$$P_b = P_1 \int_{R_2} f(r_1, r_2 | b_i = 1) dr_1 dr_2 + P_2 \left(\int_{R} f(r_1, r_2 | b_i = 0) dr_1 dr_2 - \int_{R_2} f(r_1, r_2 | b_i = 0) dr_1 dr_2 \right)$$

$$P_b = P_2 \int_{R} f(r_1, r_2 | b_i = 0) dr_1 dr_2$$

$$+ \int_{R_2} \{ P_1 f(r_1, r_2 | b_i = 1) - P_2 f(r_1, r_2 | b_i = 0) \} dr_1 dr_2$$



The first term is a constant. So, to minimize P_b assign to R_2 all values of (r_1, r_2) that make the integrand negative. That is,

Decide 0 when $P_1 f(r_1, r_2 | b_i = 1) < P_2 f(r_1, r_2 | b_i = 0)$

Decide 1 when $P_1 f(r_1, r_2 | b_i = 1) > P_2 f(r_1, r_2 | b_i = 0)$

$$\frac{f(r_1, r_2 | b_i = 1)}{f(r_1, r_2 | b_i = 0)} > \frac{P_2}{P_1},$$

Decide 1, otherwise decide 0

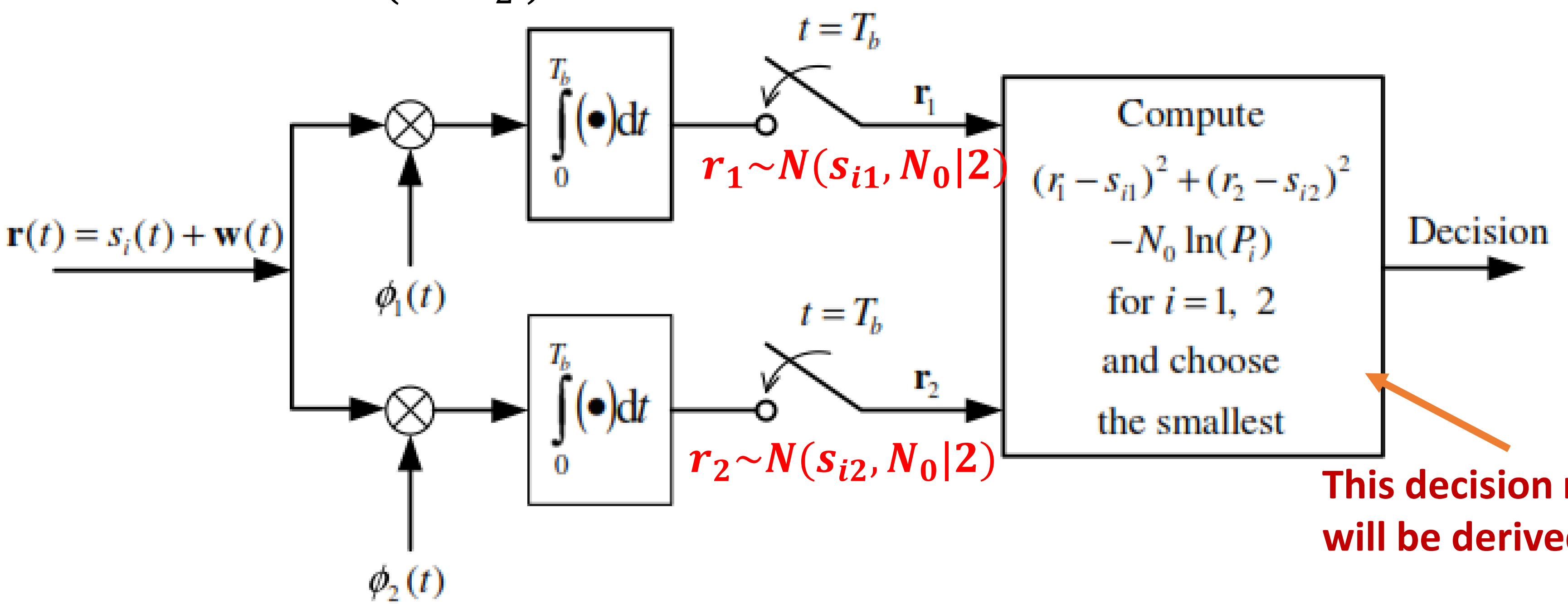
This is known as the likelihood ratio test

Receiver Structure

- The probability of error is minimized when the following decision rule is employed:
- Decide 1 $when$ $\frac{f(r_1,r_2|b_i=1)}{f(r_1,r_2|b_i=0)} \geq \frac{P_2}{P_1}$; (1); else, decide 0;
- $f(r_1, r_2|b_i = 1) = f(r_1\,|b_i = 1)f(r_2|b_i = 1)$; due to independence ; $X \sim N(\mu, \sigma^2 = N_0\,|2)$; $\frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{(x-\mu)^2}{2\pi\sigma^2}}$
- $f(r_1, r_2|b_i = 0) = f(r_1\,|b_i = 0)f(r_2|b_i = 0)$; due to independence
- $f(r_1\,|b_i = 1) \sim N\left(s_{11}, \frac{N_0}{2}\right)$; ; $f(r_2\,|b_i = 1) \sim N\left(s_{12}, \frac{N_0}{2}\right)$;
- $f(r_1\,|b_i = 0) \sim N\left(s_{21}, \frac{N_0}{2}\right)$; ;$f(r_2\,|b_i = 0) \sim N\left(s_{22}, \frac{N_0}{2}\right)$;

If s_1 is sent, then
$r_1 \sim N(s_{11}, N_0|2)$;
$r_2 \sim N(s_{12}, N_0|2)$;

If s_2 is sent, then
$r_1 \sim N(s_{21}, N_0|2)$;
$r_2 \sim N(s_{22}, N_0|2)$;



This decision rule will be derived later.

Optimum Receiver: Binary Case

$$(r_1 - s_{11})^2 + (r_2 - s_{12})^2 \underset{0_D}{\overset{1_D}{\gtrless}} (r_1 - s_{21})^2 + (r_2 - s_{22})^2 + N_0 \ln\left(\frac{P_1}{P_2}\right)$$

- For the special case of $P_1 = P_2$ (signals are equally likely):

$$(r_1 - s_{11})^2 + (r_2 - s_{12})^2 \underset{0_D}{\overset{1_D}{\gtrless}} (r_1 - s_{21})^2 + (r_2 - s_{22})^2.$$

This is the minimum distance decision rule

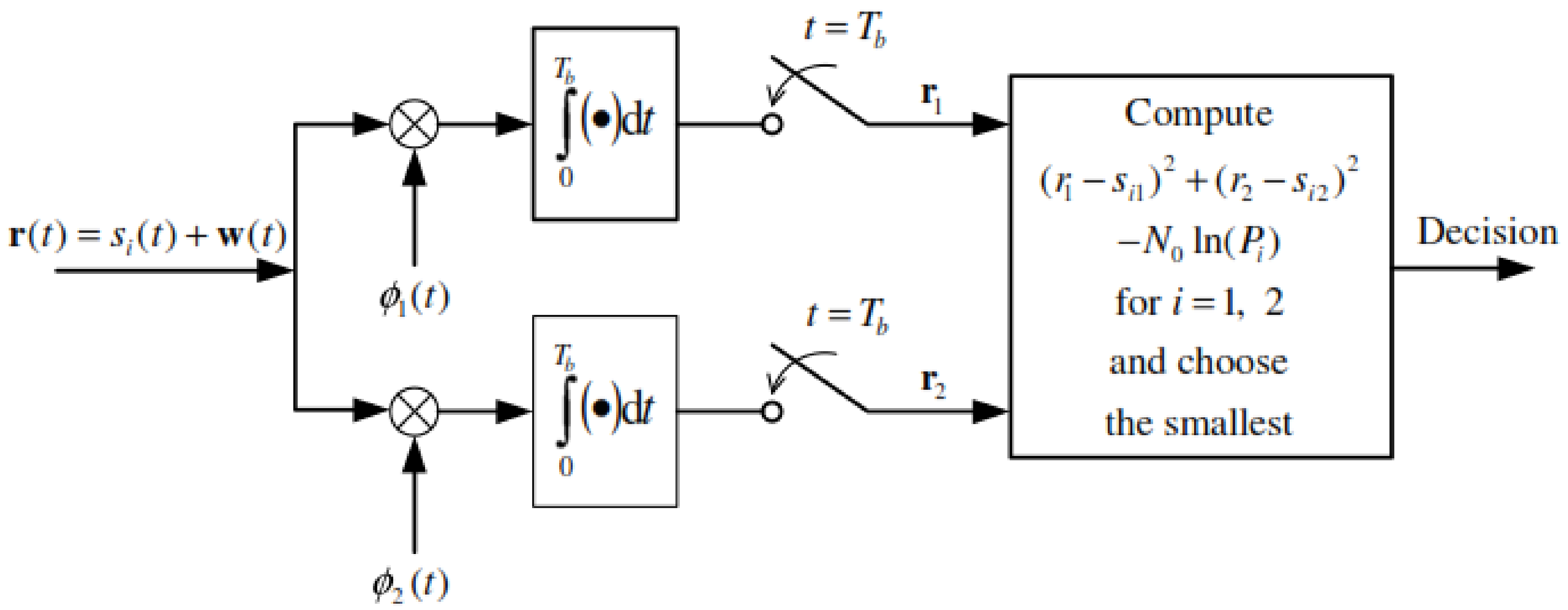
The probability of error for Equally-probable signals is given by:

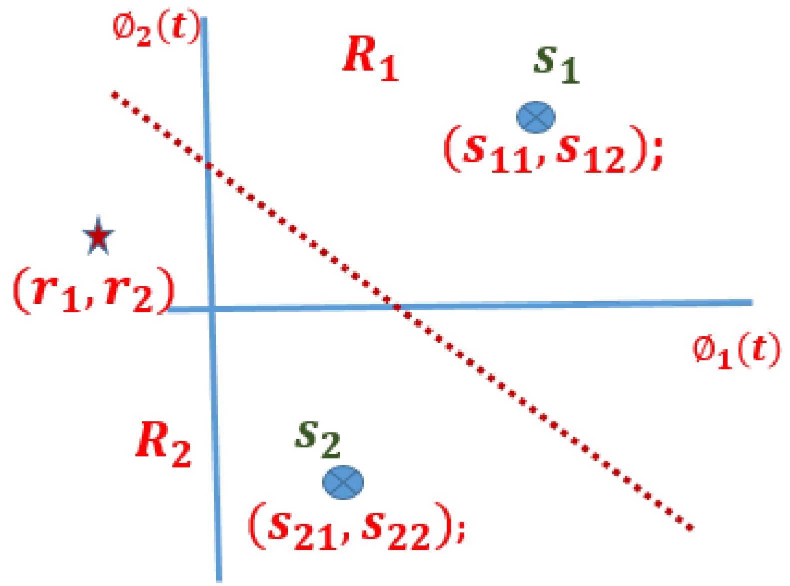
$$P_b^* = Q\left(\frac{d_{12}}{\sqrt{2N_0}}\right) = Q\left(\sqrt{\frac{\int_0^{T_b}\left(s_1(t) - s_2(t)\right)^2 dt}{2N_0}}\right)$$

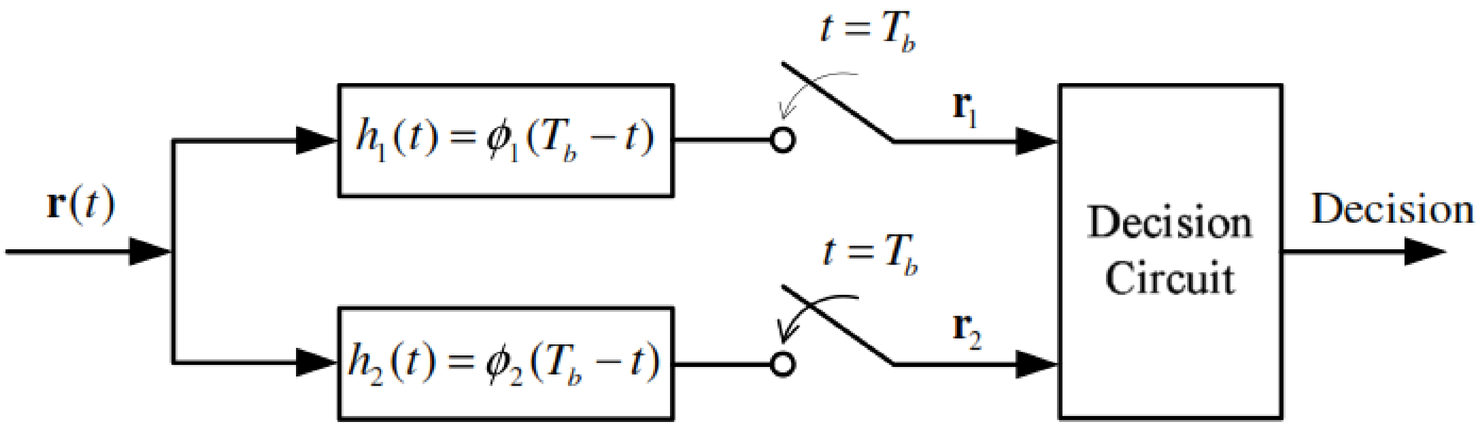
As derived earlier

$$d_{12}^2 = \int_0^{T_b} (s_1(t) - s_2(t))^2 dt = (s_{11} - s_{21})^2 + (s_{12} - s_{22})^2$$

Optimum Receiver : Matched Filter and Correlators



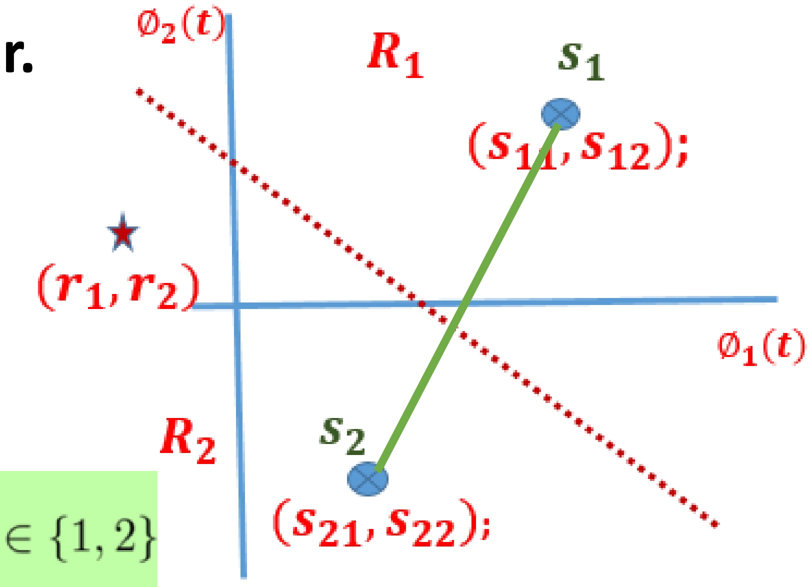




The receiver can be implemented in terms of correlators and can, as well, be implemented in terms of the matched filters. Here, matched means that the filters at the receiver are matched to the basis functions used in the transmission process. The two figures on this slide are equivalent in terms of performance.

Summary of Results on the Binary Case

Minimum Distance rule that minimizes the probability of error.



Calculate: $d_1^2 = (r_1 - s_{11})^2 + (r_2 - s_{12})^2$

Calculate: $d_2^2 = (r_1 - s_{21})^2 + (r_2 - s_{22})^2$

$$\textit{Choose } s_1 \textit{ if } d_1^2 < d_2^2$$

$$P_b^* = Q\left(\frac{d_{12}}{\sqrt{2N_0}}\right) = Q\left(\sqrt{\frac{\int_0^{T_b}(s_1(t) - s_2(t))^2 dt}{2N_0}}\right)$$

$$s_{ij} = \int_0^{T_b} s_i(t)\phi_j(t)\mathrm{d}t, \quad i, j \in \{1, 2\}$$

If s_1 is sent, then

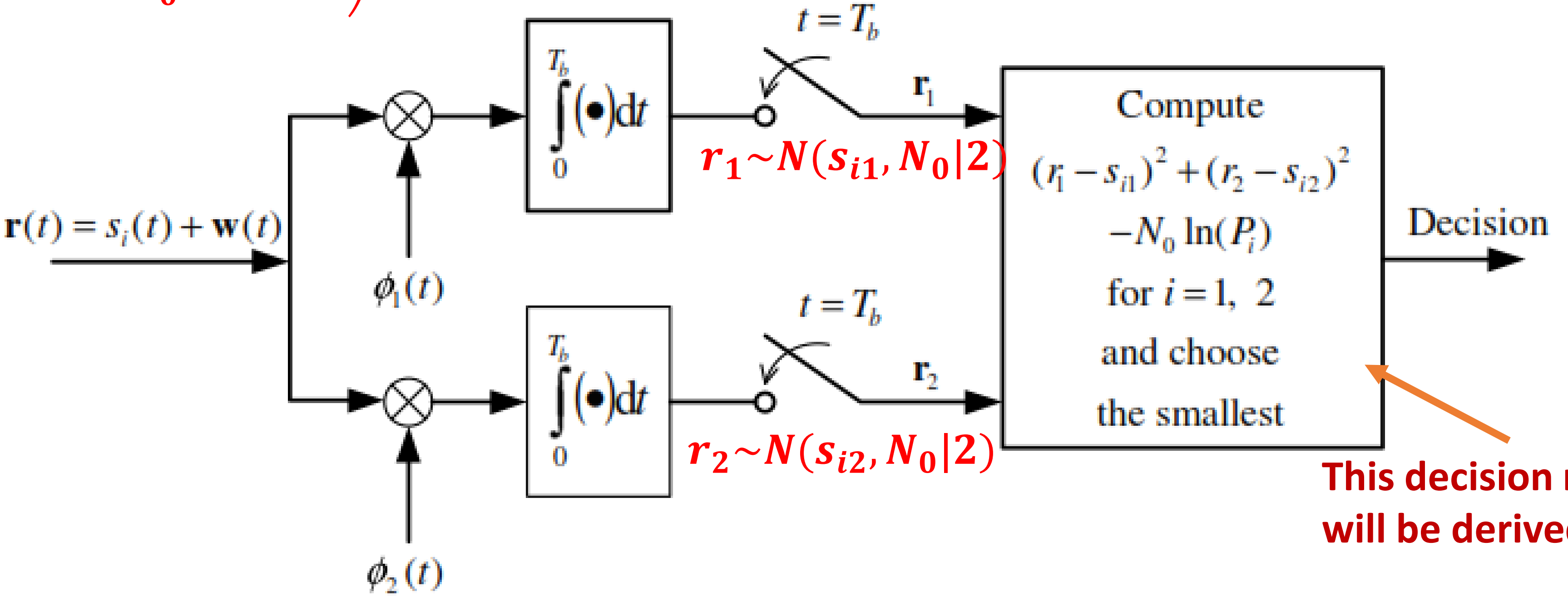
$r_1 \sim N(s_{11}, N_0|2);$

$r_2 \sim N(s_{12}, N_0|2);$

If s_2 is sent, then

$r_1 \sim N(s_{21}, N_0|2);$

$r_2 \sim N(s_{22}, N_0|2);$



This decision rule will be derived later.

M-Ary Transmission

- In M-ary transmission, a block of n binary digits are grouped together to form one symbol (message).
- If T_b is the bit duration, then $T_s = nT_b$ is the symbol duration. The data rates are related by: $R_s = R_b/n$.
- There are $M = 2^n$ possible symbols. Hence, we need $M = 2^n$ signals to be transmitted.
- The signals can modulate a high frequency carrier in the amplitude, the phase, frequency, and both the amplitude and the phase.
- We will study the following modulation techniques:
- M-ary ASK, M-ary PSK, M-ary FSK, and Quadrature Amplitude Modulation (QAM).
- For each modulation scheme, we will consider the transmitter, the optimum receiver, the probability of error, the power spectral density and the bandwidth.

M-Ary Transmission

- In most of our analysis here, we will encounter M-ary transmission in a two-dimensional space (except for the M-ary FSK), in which we need two basis functions $\emptyset_1(t)$ and $\emptyset_2(t)$.
- In this space, the signals are represented as:
- $s_1(t) = s_{11}\,\emptyset_1(t) + s_{12}\,\emptyset_2(t)$ $\qquad$ $s_2(t) = s_{21}\,\emptyset_1(t) + s_{22}\,\emptyset_2(t)$
- $s_3(t) = s_{31}\,\emptyset_1(t) + s_{32}\,\emptyset_2(t)$ $\qquad$ $s_M(t) = s_{M1}\,\emptyset_1(t) + s_{M2}\,\emptyset_2(t)$
- **Where** $s_{i1} = \int_0^{T_s} s_i(t)\,\emptyset_1(t)dt,$ $\qquad$ $s_{i2} = \int_0^{T_s} s_i(t)\,\emptyset_2(t)dt$
- The receiver has to decide on which signal was transmitted based on the received vector (r_1, r_2).
- Note: To obtain the basis functions from the M given functions, one can use the **Gram Schmidt orthogonalization procedure. The number of basis functions N<=M.**

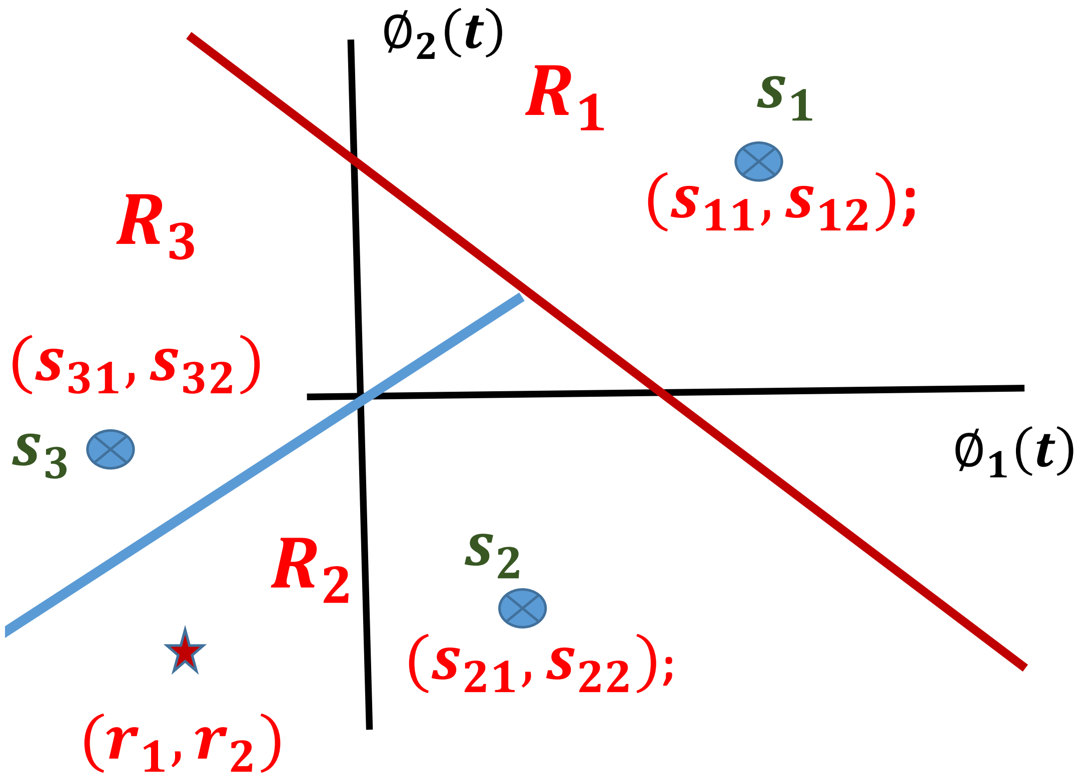
Optimum Receiver for M-Ary Transmission

- **The observation space is to be partitioned into M regions, such that if the set of measurements fall into region R_k signal s_k is declared true.**
- **It is assumed here that all signals are equally probable.**
- **The receiver collects the measurements from the N correlators (r vector) and calculates the distance to each of the N signals.**
- **It decides in favor of the signal closest to the (r vector).**

Minimum Distance Rule

$$Choose\ s_1\ if\ d_i^2 < d_k^2;\ for\ all\ k\ signals$$

$$\sum_{k=1}^{N}(r_k - s_{ik})^2 < \sum_{k=1}^{N}(r_k - s_{jk})^2;$$
$$j = 1, 2, \ldots, M;\ j \neq i.$$



M-ary Coherent Amplitude-Shift Keying (M-ASK)

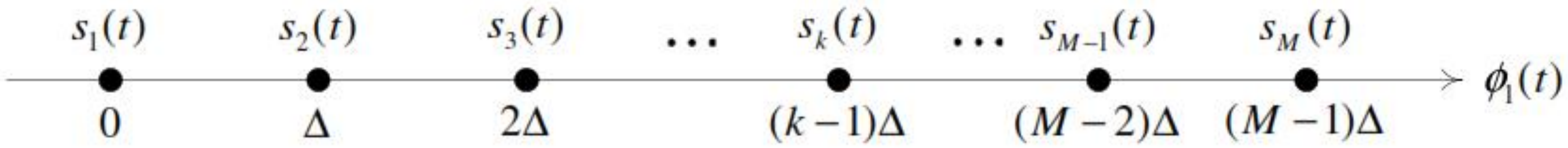
$$s_i(t) = V_i\sqrt{\frac{2}{T_s}}\cos(2\pi f_c t),\ 0 \le t \le T_s \qquad \boldsymbol{s_i(t) = V_i\phi_1(t)}$$

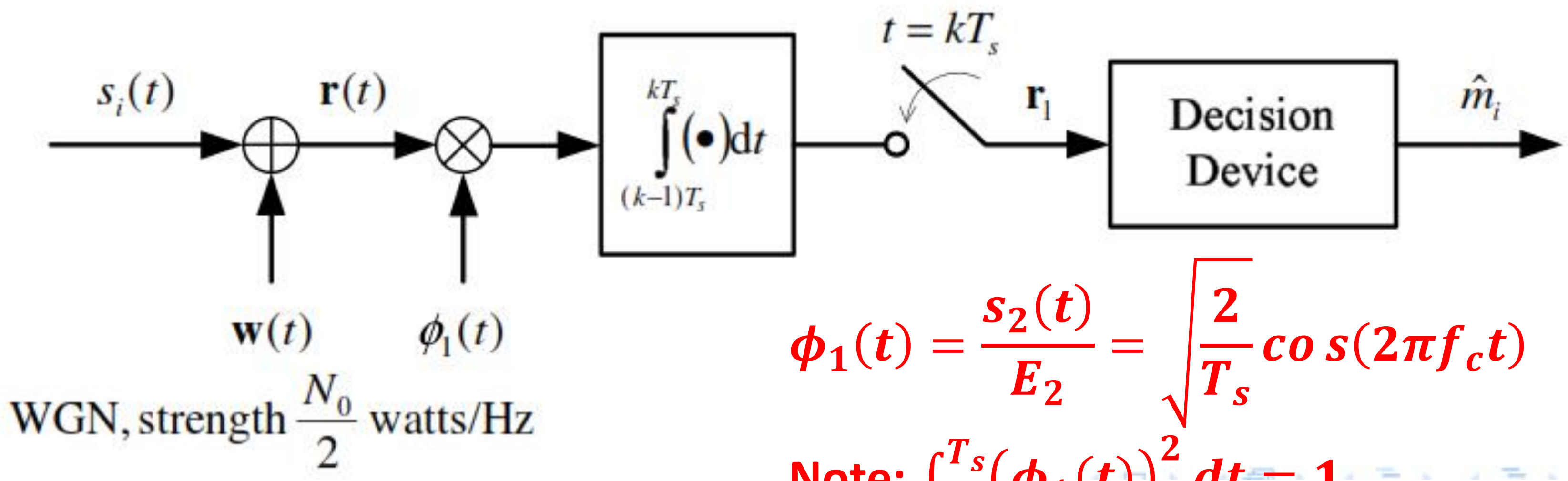
$$\boldsymbol{V_i = (i-1)\Delta}$$

$$= [(i-1)\Delta]\phi_1(t), \quad \phi_1(t) = \sqrt{\frac{2}{T_s}}\cos(2\pi f_c t),\ 0 \le t \le T_s,$$

$$i = 1, 2, \ldots, M. \qquad \boldsymbol{E_i = ((i-1)\Delta)^2} \qquad \boldsymbol{E_i = (V_i)^2}$$

In this case, we have M signals. However, we need only one base function. The signals are linearly dependent and hence, every signal can be expressed in terms of this base function.





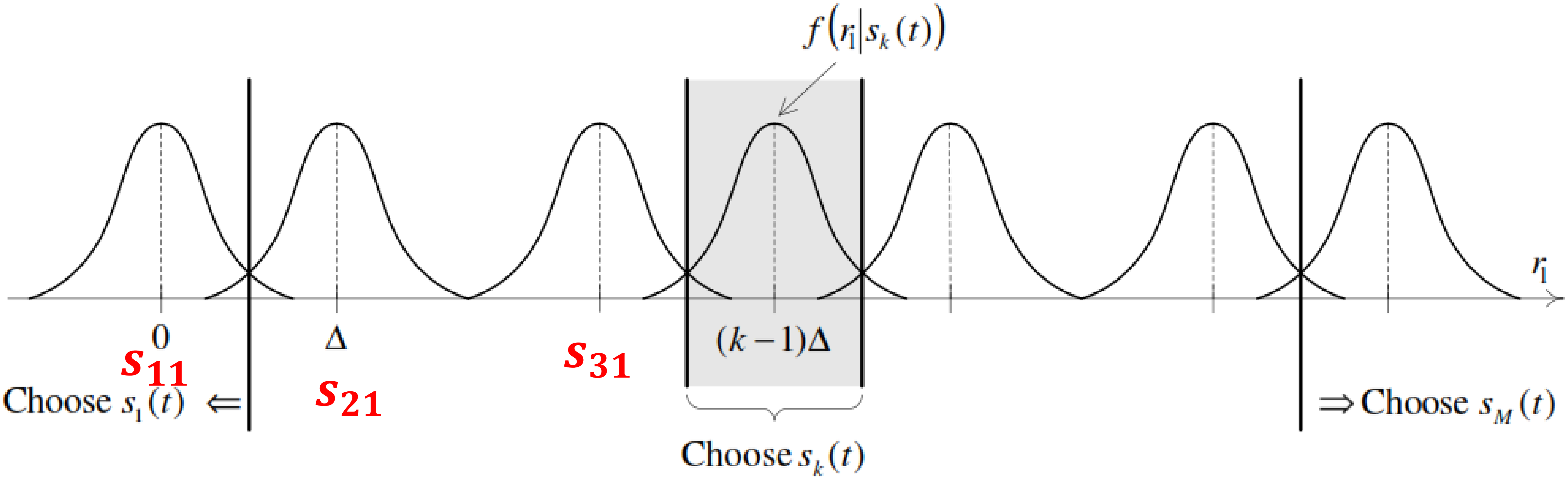
$$\boldsymbol{\phi_1(t) = \frac{s_2(t)}{E_2} = \sqrt{\frac{2}{T_s}}\cos(2\pi f_c t)}$$

Note: $\boldsymbol{\int_0^{T_s}(\phi_1(t))^2\,dt = 1}$

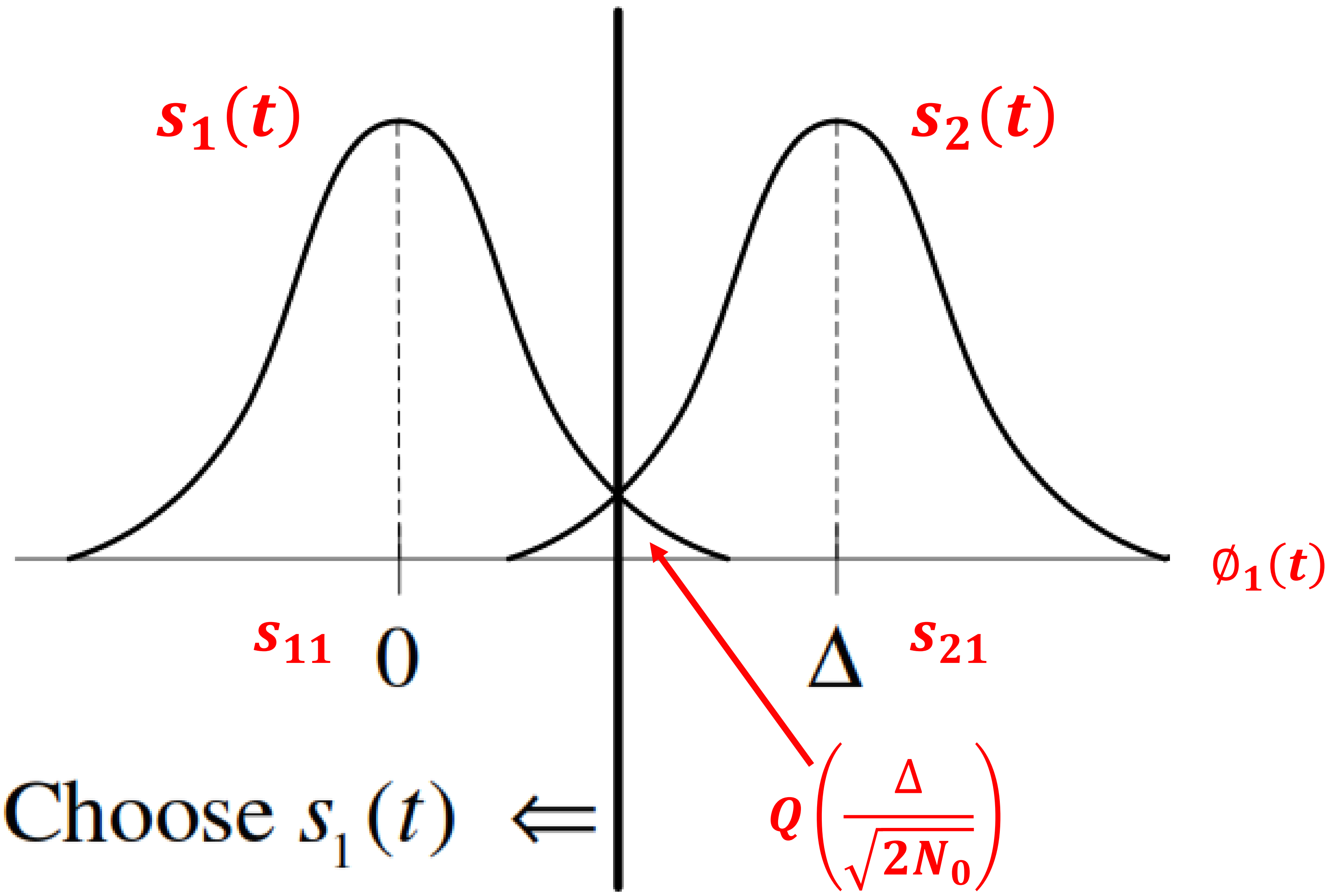
Since there is one base function, the receiver consists of one correlator (multiplier followed by an integrator), a sampler, and a decision device (set of comparators).

Minimum-Distance Decision Rule for M-ASK

$$\text{Choose}\begin{cases} s_k(t), & \text{if} \quad \left(k-\frac{3}{2}\right)\Delta < r_1 < \left(k-\frac{1}{2}\right)\Delta,\ k = 2,3,\ldots,M-1 \\ s_1(t), & \text{if} \quad r_1 < \frac{\Delta}{2} \\ s_M(t), & \text{if} \quad r_1 > \left(M-\frac{3}{2}\right)\Delta \end{cases}$$



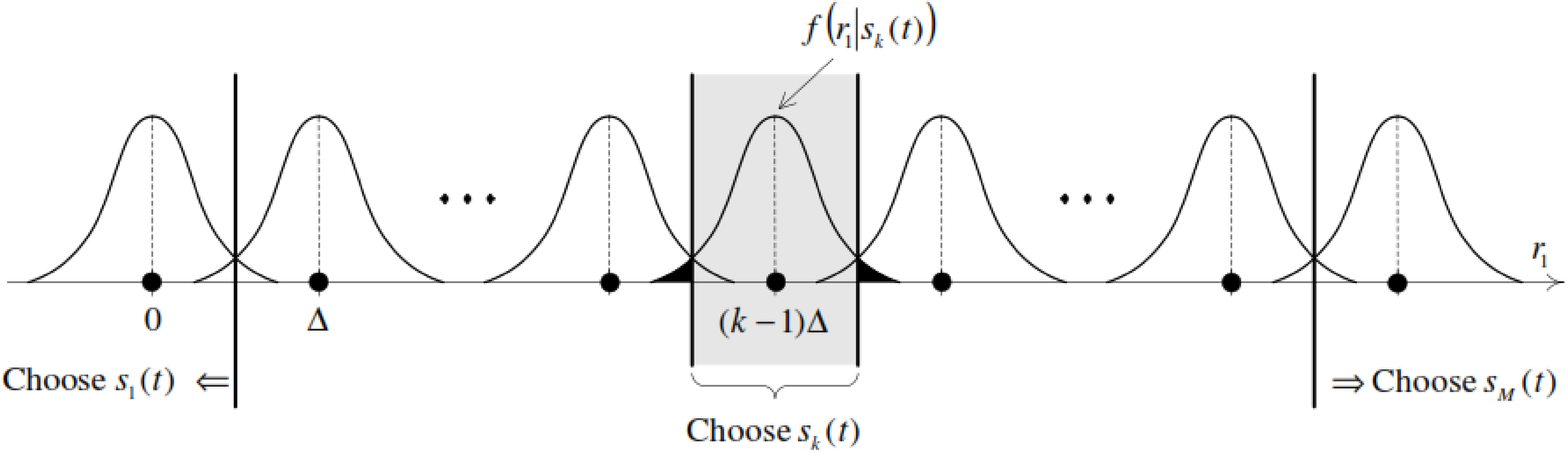
Minimum Distance Rule and Error Probability for two signals



$$P_b^* = Q\left(\frac{d_{12}}{\sqrt{2N_0}}\right)$$

$$P_b^* = Q\left(\frac{\Delta}{\sqrt{2N_0}}\right)$$

Minimum-Distance Decision and Error Probability for M-ASK



$$P[\text{error}] = \sum_{i=1}^{M} P[s_i(t)]P[\text{error}|s_i(t)]$$

$$P[\text{error}|s_i(t)] = 2Q\left(\Delta/\sqrt{2N_0}\right), \quad i = 2, 3, \ldots, M-1$$

$$P[\text{error}|s_i(t)] = Q\left(\Delta/\sqrt{2N_0}\right), \quad i = 1, M$$

$$P[\text{error}] = \frac{2(M-1)}{M} Q\left(\Delta/\sqrt{2N_0}\right).$$

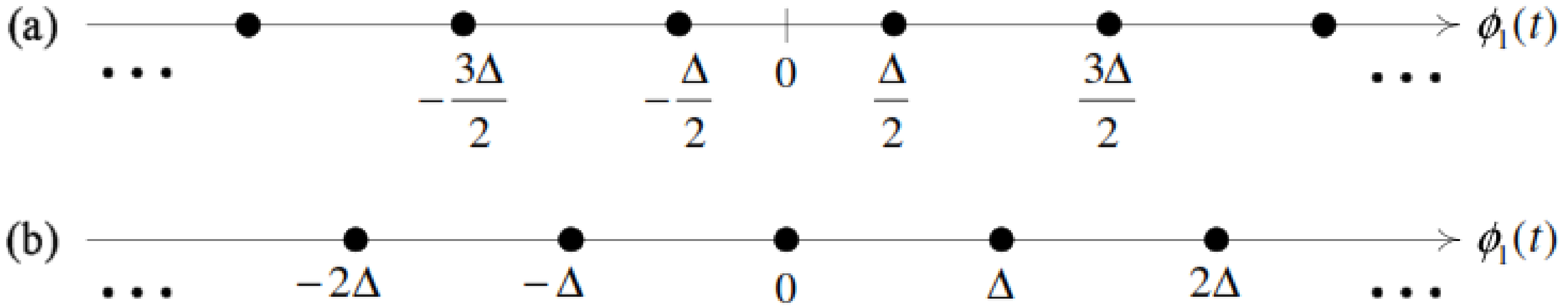
For a given M, $P[\text{error}]$ depends on the noise power (N_0) and the *minimum* distance δ. This means that moving the origin of the signal constellation does not affect the performance!

Modified M-ASK Constellation

The maximum and average transmitted energies can be reduced, without any sacrifice in error probability, by changing the signal set to one which includes the negative version of each signal.

$$E_i = (V_i)^2$$

$$s_i(t) = \underbrace{(2i-1-M)\frac{\Delta}{2}}_{V_i}\sqrt{\frac{2}{T_s}}\cos(2\pi f_c t),\ 0 \le t \le T_s,\ i = 1, 2, \ldots, M.$$



$$E_s = \frac{\sum_{i=1}^{M} E_i}{M} = \frac{\Delta^2}{4M}\sum_{i=1}^{M}(2i-1-M)^2 = \frac{(M^2-1)\Delta^2}{12}.$$

$$E_b = \frac{E_s}{\log_2 M} = \frac{(M^2-1)\Delta^2}{12\log_2 M} \Rightarrow \Delta = \sqrt{\frac{(12\log_2 M)E_b}{M^2-1}}$$

Es: Average Energy per Symbol

Eb: Average Energy per bit

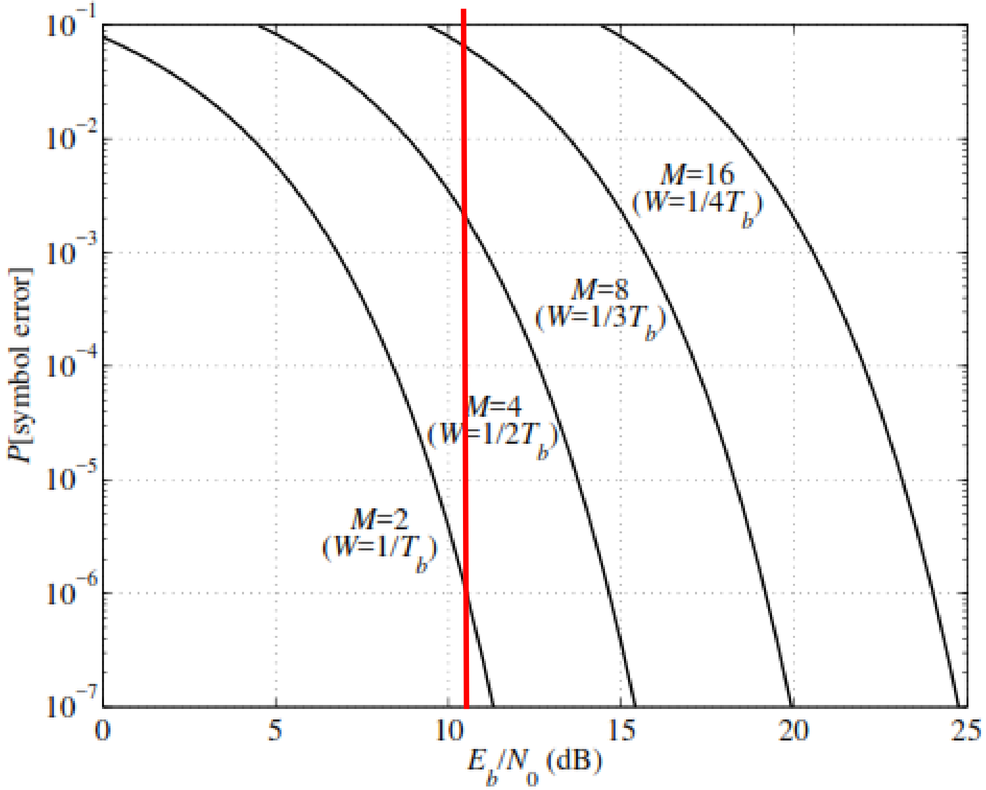
Probability of Symbol Error for M-ASK

$$P[\text{error}] = \frac{2(M-1)}{M} Q\left(\sqrt{\frac{6E_s}{(M^2-1)N_0}}\right) = \frac{2(M-1)}{M} Q\left(\sqrt{\frac{6\log_2 M}{M^2-1}\frac{E_b}{N_0}}\right).$$

Symbol error probability

$$P[\text{bit error}] = \frac{1}{\lambda} P[\text{symbol error}] = \frac{2(M-1)}{M\log_2 M} Q\left(\sqrt{\frac{6\log_2 M}{M^2-1}\frac{E_b}{N_0}}\right) \quad \text{(with Gray mapping)}$$

Bit error probability



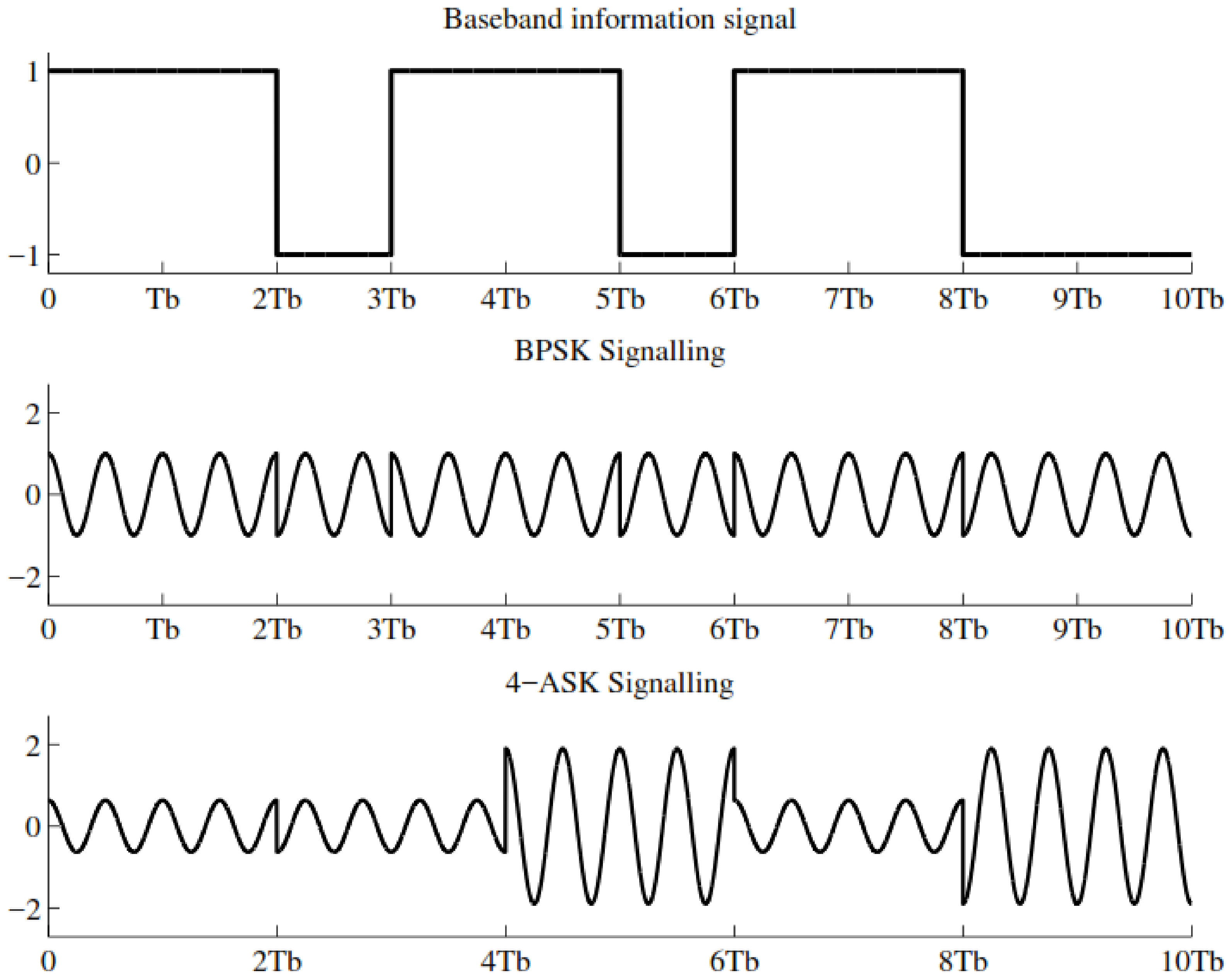
Two comments:

Error probability: for a given Eb/N0, increasing M results in an increase in the error probability.

Bandwidth: Increasing M results in a reduction in the bandwidth by a factor of $\lambda = log_2(M)$.

W is obtained by using the $WT_s = 1$ rule-of-thumb. Here $1/T_b$ is the bit rate (bits/s).

Example of 2-ASK (BPSK) and 4-ASK Signals



Binary sequence:
1101101100

$\mathbf{1 \rightarrow cos(2\pi f_0)\, t}$
$\mathbf{0 \rightarrow -cos(2\pi f_0)\, t}$
(similar to BPSK)

$\mathbf{11 \rightarrow \cos(2\pi f_0)\, t}$
$\mathbf{01 \rightarrow -cos(2\pi f_0)\, t}$
$\mathbf{10 \rightarrow 2cos(2\pi f_0)\, t}$
$\mathbf{00 \rightarrow -2\, cos(2\pi f_0)\, t}$

M-ary Phase-Shift Keying (M-PSK)

$$s_i(t) = V\cos\left[2\pi f_c t - \frac{(i-1)2\pi}{M}\right], \quad 0 \le t \le T_s,$$

$$E_i = \int_0^{T_s} (s_i(t)^2 dt = V^2 T_s/2;$$

Same for all i

$i = 1, 2, \ldots, M;\ f_c = k/T_s,\ k$ integer; $E_s = V^2 T_s/2$ joules

Note: $\int_0^{T_s} (\phi_1(t))^2 dt = 1$

$: \int_0^{T_s} (\phi_2(t))^2 dt = 1$

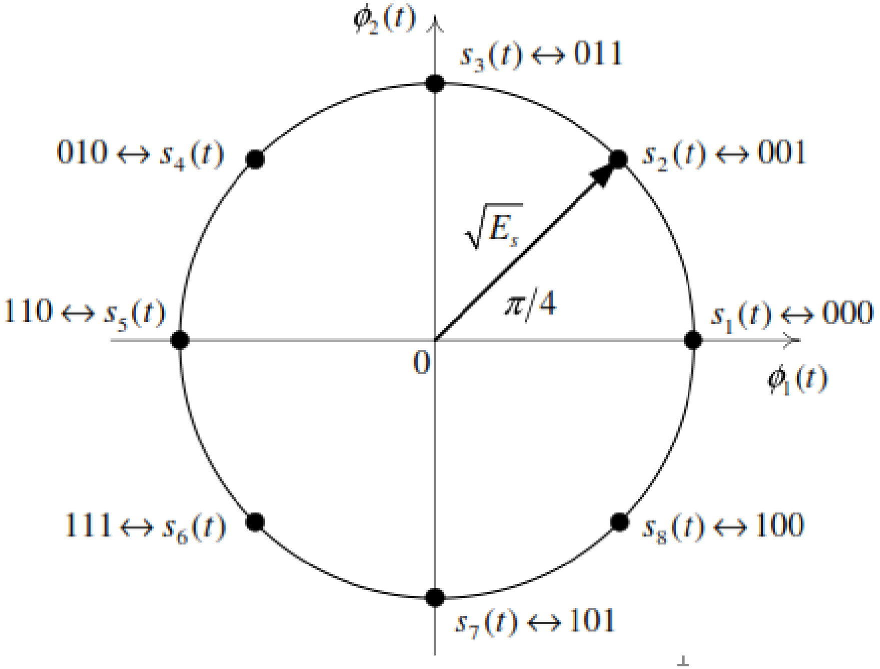
$$\int_0^{T_s} \phi_1(t)\,\phi_2(t) dt = 0$$

$$s_i(t) = V\cos\left[\frac{(i-1)2\pi}{M}\right]\cos(2\pi f_c t) + V\sin\left[\frac{(i-1)2\pi}{M}\right]\sin(2\pi f_c t).$$

$$\phi_1(t) = \frac{V\cos(2\pi f_c t)}{\sqrt{E_s}},\quad \phi_2(t) = \frac{V\sin(2\pi f_c t)}{\sqrt{E_s}}.$$

$$s_{i1} = \sqrt{E_s}\cos\left[\frac{(i-1)2\pi}{M}\right],\quad s_{i2} = \sqrt{E_s}\sin\left[\frac{(i-1)2\pi}{M}\right].$$

The signals lie on a circle of radius $\sqrt{E_s}$, and are spaced every $2\pi/M$ radians around the circle.



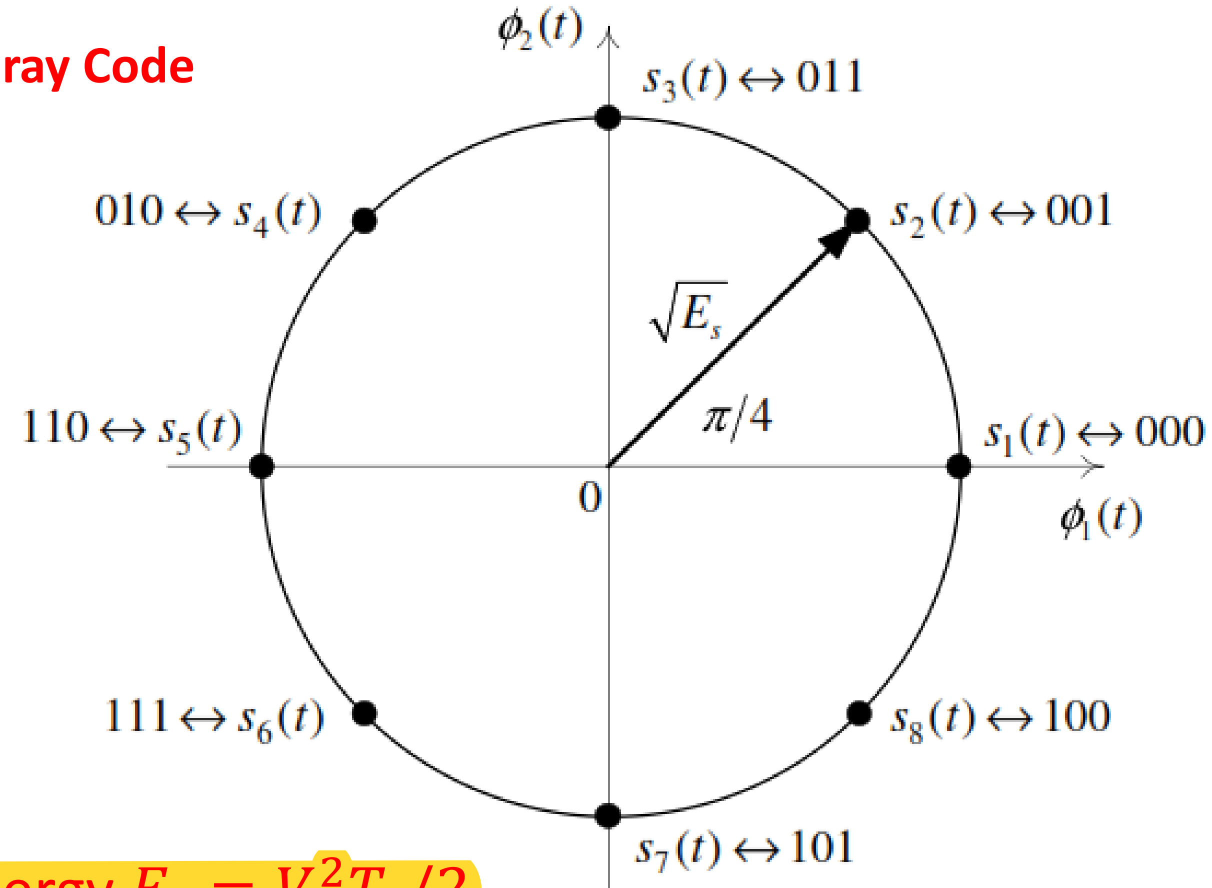
$$\phi_1(t) = \sqrt{\frac{2}{T_s}}\cos(2\pi f_c t);\ \phi_2(t) = \sqrt{\frac{2}{T_s}}\sin(2\pi f_c t)$$

M-ary Phase-Shift Keying (M-PSK)

$$s_i(t) = V\cos\left[2\pi f_c t - \frac{(i-1)2\pi}{M}\right], \quad 0 \le t \le T_s,$$

$$i = 1, 2, \ldots, M;\ f_c = k/T_s,\ k \text{ integer};\ E_s = V^2 T_s/2 \text{ joules}$$

Use Gray Code



Signal Energy $E_s = V^2 T_s/2$

- **Here, the amplitude of the carrier remains constant, however the phase takes on one of M possible values.**
- **Two base functions are needed to represent all signals in the two-dimensional signal space.**
- **The spacing between adjacent signals is $\Delta\theta = 2\pi/M$ radians.**
- **In this example, M=8 and $\Delta\theta = \frac{\pi}{4} = 45\ degrees.$**
- **To minimize error, gray coding is used.**

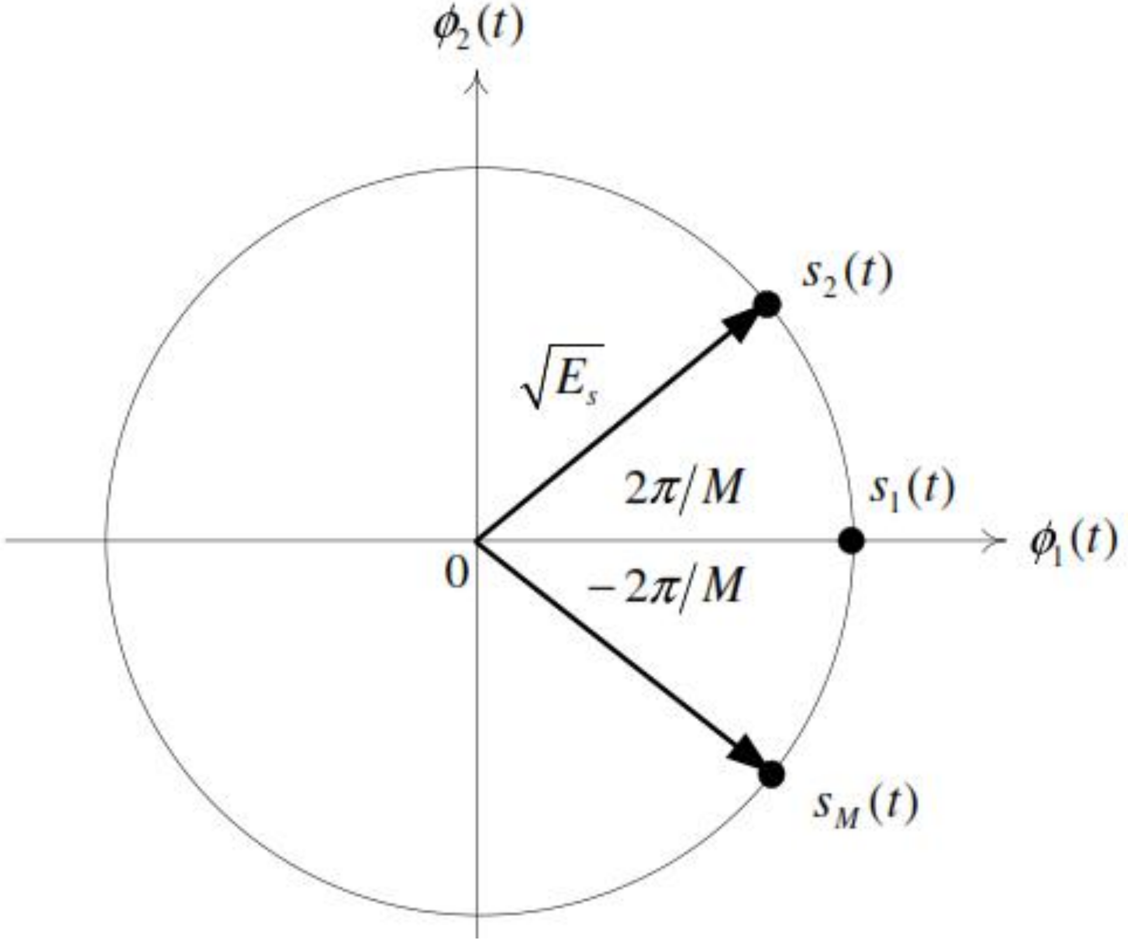
M-ary Phase-Shift Keying (M-PSK): Signal Space Representation

$$s_i(t) = V\cos\left[2\pi f_c t - \frac{(i-1)2\pi}{M}\right], \quad 0 \le t \le T_s,$$

$$i = 1, 2, \ldots, M;\ f_c = k/T_s,\ k \text{ integer};\ E_s = V^2 T_s/2 \text{ joules}$$

$$s_{i1} = \sqrt{E_s}\cos\left[\frac{(i-1)2\pi}{M}\right]$$

$$s_{i2} = \sqrt{E_s}\sin\left[\frac{(i-1)2\pi}{M}\right]$$



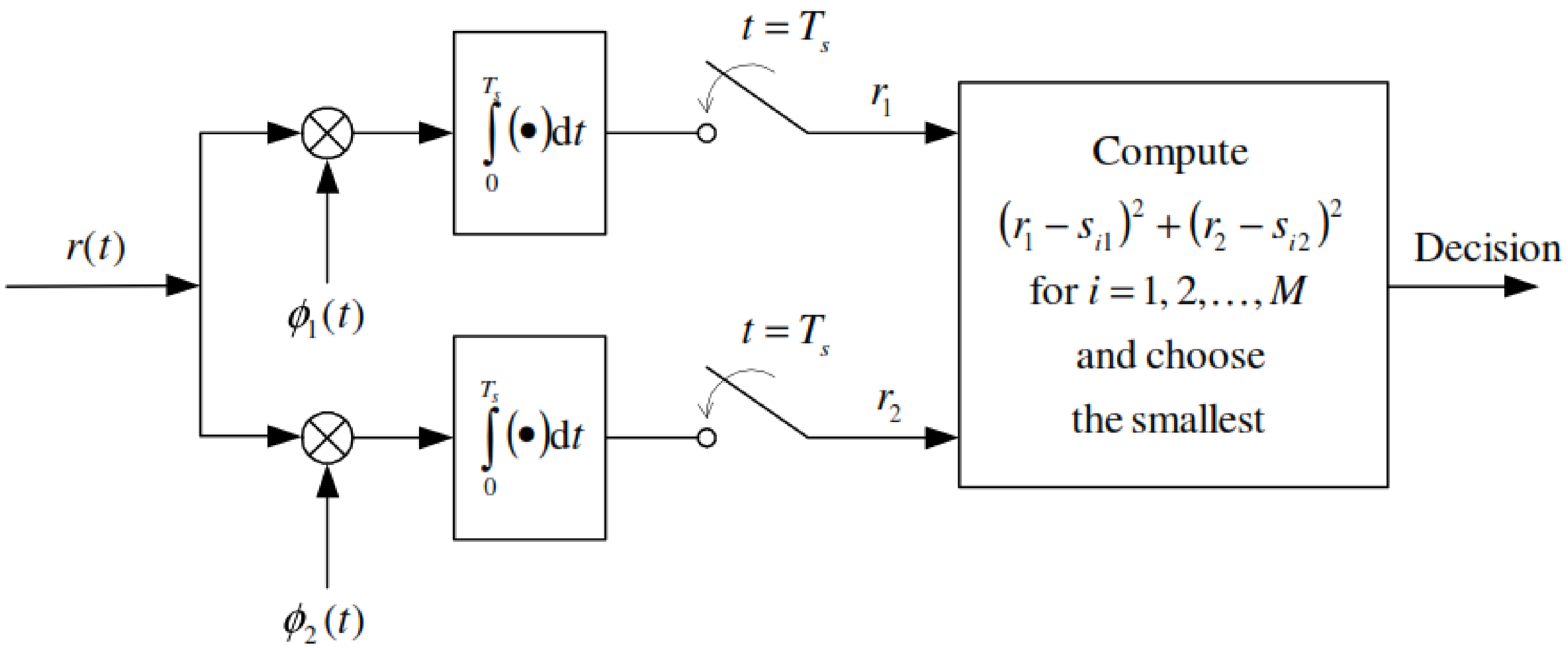
Optimum receiver in a two-dimensional space

$$r_1 = \int_0^{T_b} \left(s_i(t) + w(t)\right)\emptyset_1(t)dt$$

$$r_1 = s_{i1} + \ N_1;\ N_1 \sim N(0, N_0|2);$$

$$r_2 = \int_0^{T_b} \left(s_i(t) + w(t)\right)\emptyset_2(t)dt$$

$$r_1 = s_{i2} + \ N_2;\ N_2 \sim N(0, N_0|2);$$



$r_1 \sim N(s_{i1}, N_0|2)$; Gaussian with mean s_{i1}, variance $N_0|2$

$r_2 \sim N(s_{i2}, N_0|2)$; Gaussian with mean s_{i2}, variance $N_0|2$

r_1 *and* r_2 **are independent**

Minimum Distance Rule

Calculate: $d_1^2 = (r_1 - s_{11})^2 + (r_2 - s_{12})^2$

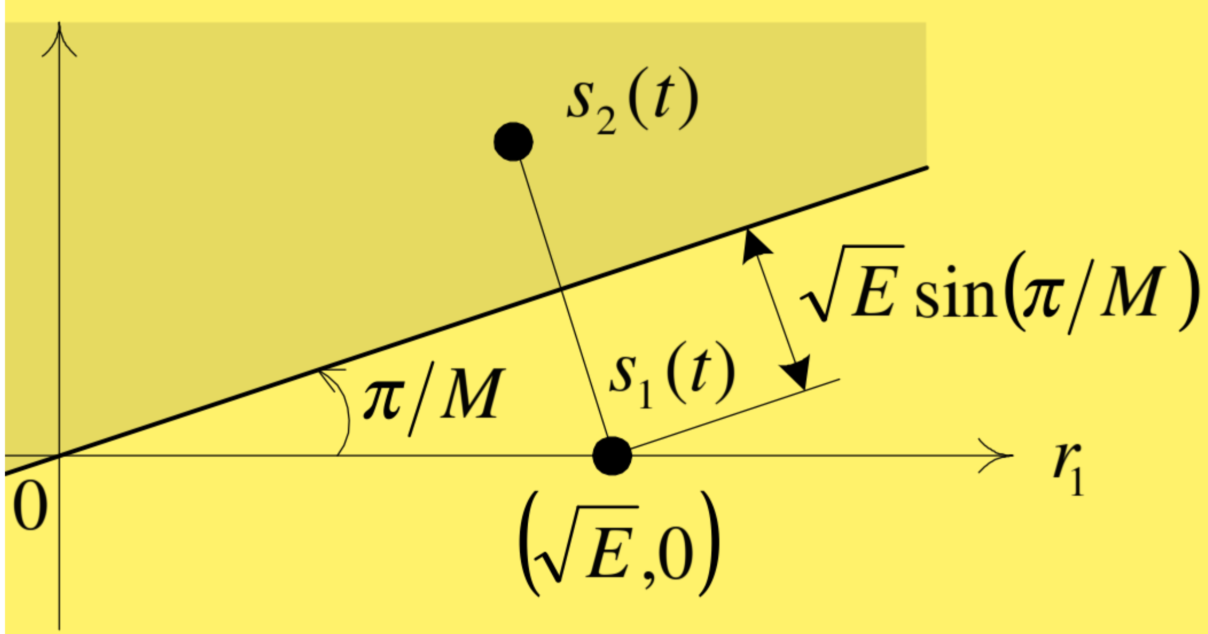
Calculate: $d_2^2 = (r_1 - s_{21})^2 + (r_2 - s_{22})^2$

Choose s_1 *if* $d_1^2 < d_2^2$

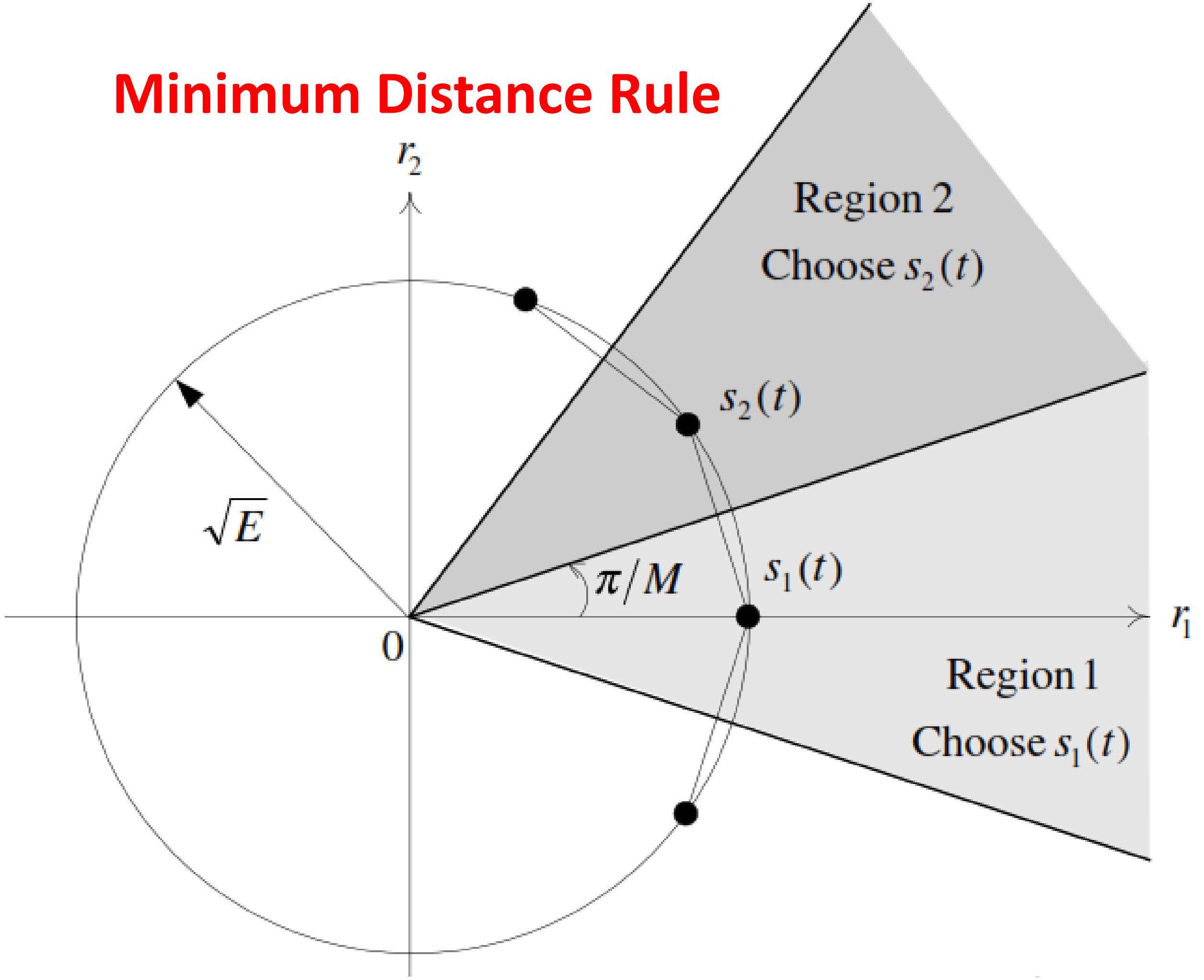
M-ary Phase-Shift Keying: Error Probability

$$\begin{aligned}\Pr[\text{error}] &= \Pr[\text{error}|s_1(t)] = \Pr[r_1, r_2 \text{ fall outside Region } 1|s_1(t) \text{ transmitted}] \\ &= 1 - \Pr[r_1, r_2 \text{ fall in Region } 1|s_1(t) \text{ transmitted}] \\ &= 1 - \iint_{r_1, r_2 \in \mathcal{R}_1} f(r_1, r_2|s_1(t))\mathrm{d}r_1 \mathrm{d}r_2\end{aligned}$$

Minimum Distance Rule



$$d_{\min} = 2\sqrt{E_s}\sin\left(\frac{\pi}{M}\right)$$

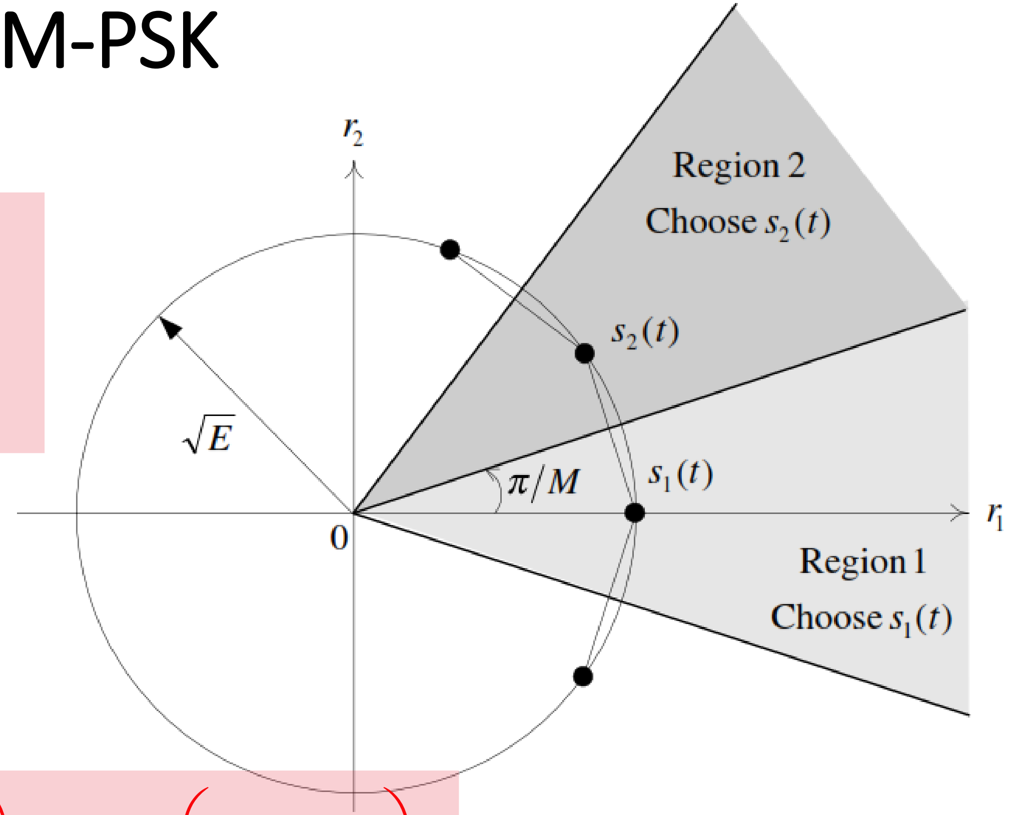


Probability of Error in M-PSK

- **The distance between two neighboring symbols is**

$$d_{\min} = 2\sqrt{E_s}\sin\left(\frac{\pi}{M}\right) \qquad P_b^* = Q\left(\frac{d_{12}}{\sqrt{2N_0}}\right)$$

- **Each symbol has 2 close neighbor symbols.**
- **An approximation for the symbol error prob.**



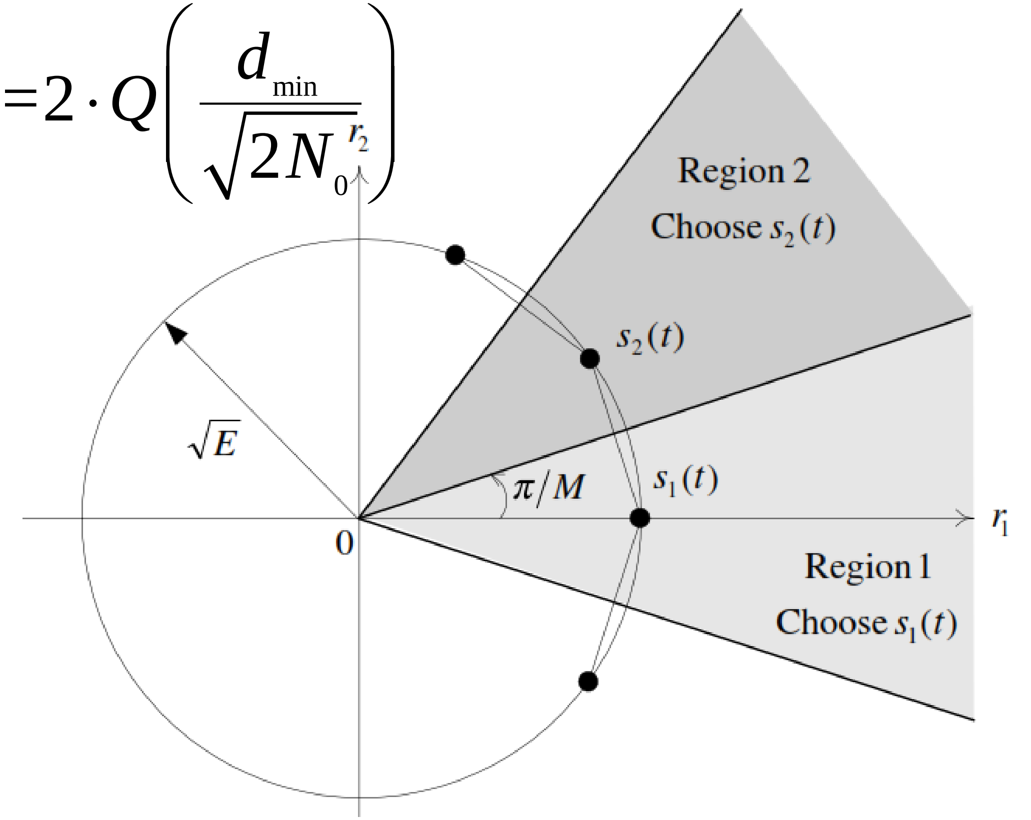
$$\mathrm{P_s} \approx (\text{Number of Signals at distance dmin})\ Q\left(\frac{d_{\min}}{\sqrt{2N_0}}\right) = 2\cdot Q\left(\frac{d_{\min}}{\sqrt{2N_0}}\right)$$

$$= 2\cdot Q\left(\frac{2\sqrt{E_s}\sin\left(\frac{\pi}{M}\right)}{\sqrt{2N_0}}\right) = 2\cdot Q\left(\sqrt{2\frac{E_s}{N_0}\sin^2\left(\frac{\pi}{M}\right)}\right)$$

Probability of Error in M-PSK

$$P_s \approx \text{(Number of Signals at distance dmin) } Q\left(\frac{d_{\min}}{\sqrt{2N_0}}\right) = 2\cdot Q\left(\frac{d_{\min}}{\sqrt{2N_0}}\right)$$

$$= 2\cdot Q\left(\frac{2\sqrt{E_s}\sin\left(\frac{\pi}{M}\right)}{\sqrt{2N_0}}\right) = 2\cdot Q\left(\sqrt{2\frac{E_s}{N_0}\sin^2\left(\frac{\pi}{M}\right)}\right)$$



$$P_s \approx 2Q\left(\sqrt{\frac{E_s}{N_0}}\right);\ \text{QPSK; M=4}$$

$$P_s \approx 2Q\left(\sqrt{0.293\frac{E_s}{N_0}}\right);\ \text{for 8-PSK}$$

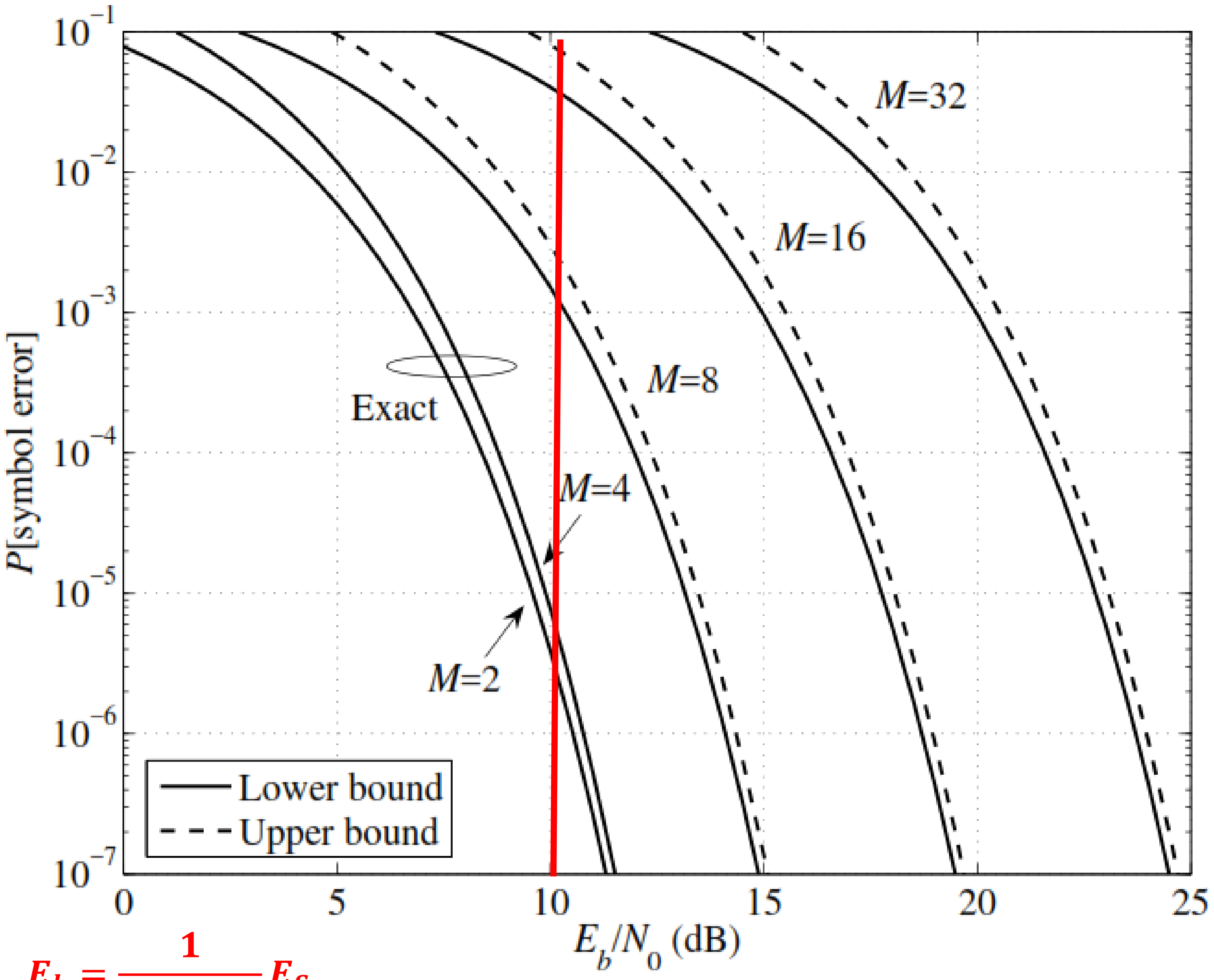
$$P_s \approx 2Q\left(\sqrt{0.038\frac{E_s}{N_0}}\right);\ \text{for16-PSK}$$

Note that as M increases, the symbol error probability increases for the same symbol energy

Symbol and Bit Error Probability of M-PSK

- When Gray coding is used, the symbol and bit error probabilities are related by: $\boldsymbol{P_b = \frac{1}{log_2(M)} P_S}$;
- Moreover, the symbol energy is related to the bit energy by
- $\boldsymbol{E_b = \frac{1}{log_2(M)} E_S}$
- The performance of digital communication systems is usually taken as the error probability versus $\frac{E_b}{N_0}$.
- The next figure depicts the symbol probability of error for M-PSK

Performance of M-PSK



$$E_b = \frac{1}{log_2(M)} E_S$$

$$P_s \approx 2 \cdot Q\left(\sqrt{2\frac{E_s}{N_0}\sin^2\left(\frac{\pi}{M}\right)}\right)$$

As M increases, the symbol probability of error increases. Note that as M increases, the spacing between signals around the perimeter of the unit circle becomes smaller, and this results in a higher probability of error

M-ary Coherent Frequency-Shift Keying (M-FSK)

Signal Set:

$$s_m(t) = \sqrt{\frac{2E_s}{T_s}}\cos(2\pi(f_c+m\Delta f)t); \quad \text{m =1, 2, …, M}, \quad 0 \le t \le T_s$$

Orthogonality condition:

$$\int_0^{T_s} s_i(t)s_j(t)\,dt = 0, i \neq j$$

The minimum frequency separation between signals to make them orthogonal

is $\Delta f = \frac{1}{2T_s} = \frac{R_s}{2}$

All signals have the same energy

$$E_s = E = \int_0^{T_s} s_m(t)^2\ dt$$

As a result of this condition, there will be M basis functions

$$\emptyset_m(t) = \frac{s_m(t)}{\sqrt{E}} = \sqrt{\frac{2}{T_s}}\cos(2\pi f_m t); \quad f_m = f_c+m\Delta f$$

M-ary Coherent Frequency-Shift Keying: Signal Space Representation

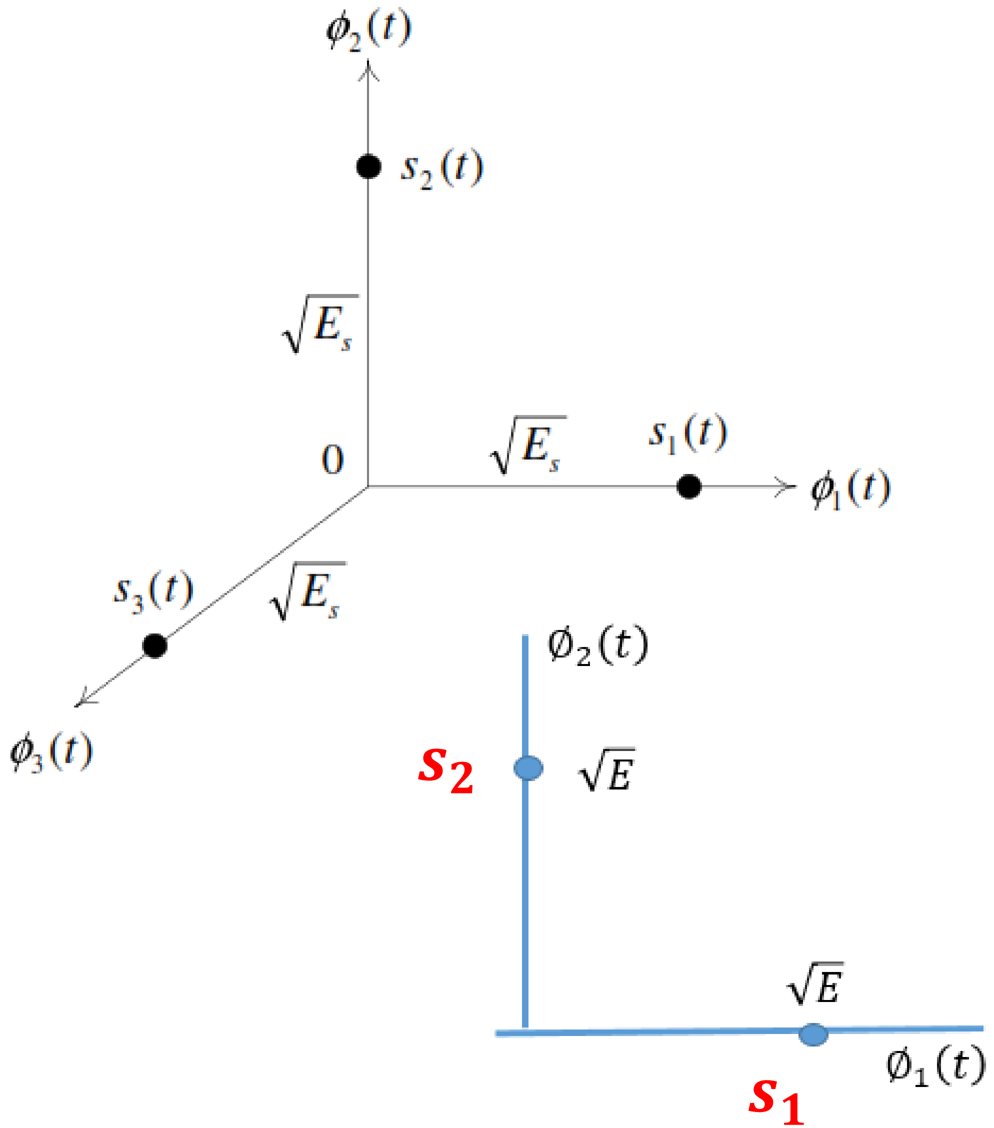
M-ary orthogonal FSK has a geometric presenation as M-dim orthogonal vectors, given as

$$\mathbf{s}_0 = \left(\sqrt{E_s}, 0, 0, \cdots, 0\right)$$

$$\mathbf{s}_1 = \left(0, \sqrt{E_s}, 0, \cdots, 0\right)$$

$$\vdots$$

$$\mathbf{s}_{M-1} = \left(0, 0, \cdots, 0, \sqrt{E_s}\right)$$



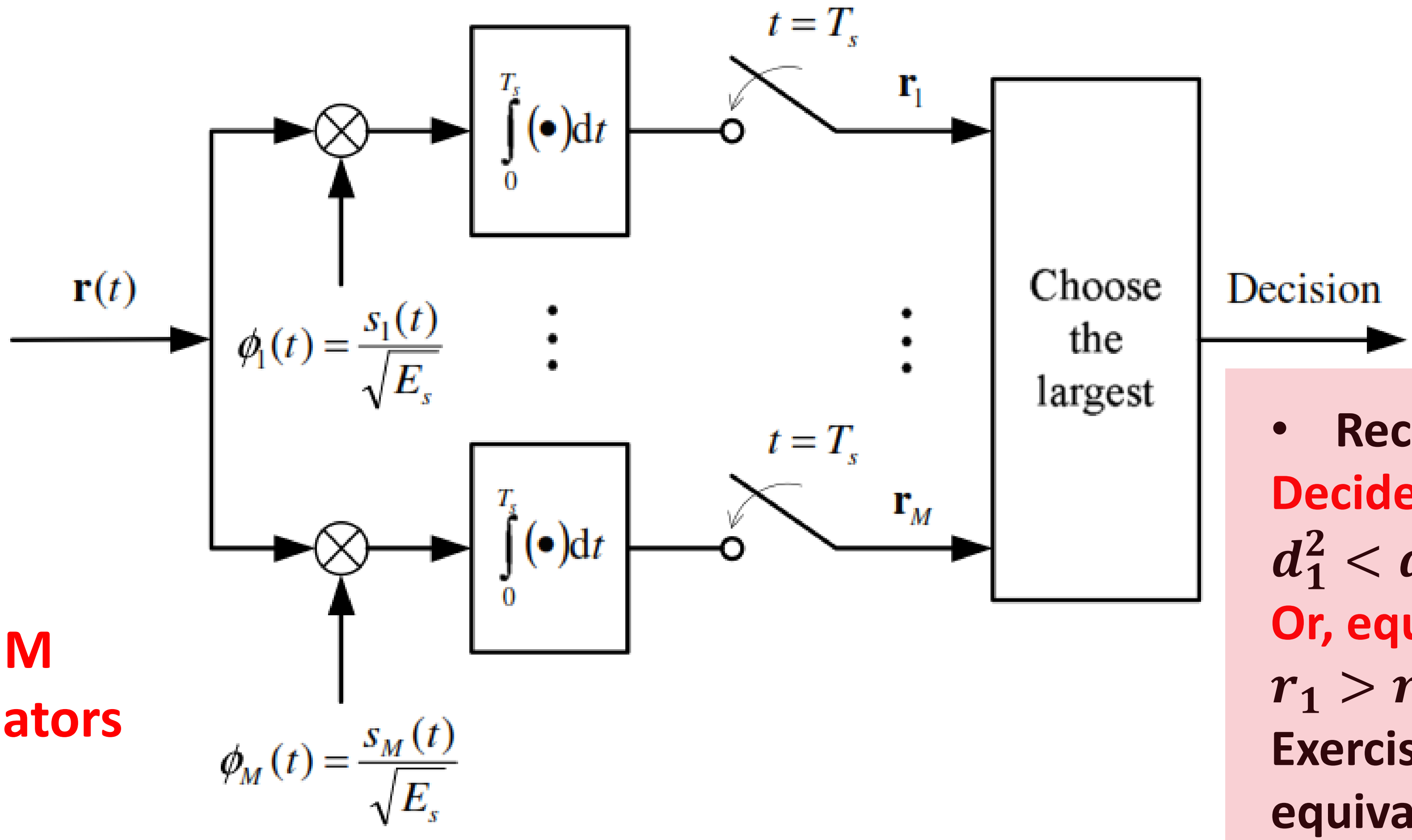
Signals are orthogonal

Minimum-Distance Receiver of M-FSK

Choose m_i if

$$\sum_{k=1}^{M}(r_k - s_{ik})^2 < \sum_{k=1}^{M}(r_k - s_{jk})^2 \quad \Rightarrow \quad \boxed{\begin{array}{c}\text{Choose } m_i \text{ if} \\ r_i > r_j, \quad j = 1, 2, \ldots, M;\ j \neq i.\end{array}}$$

$$j = 1, 2, \ldots, M;\ j \neq i,$$



- **The receiver consists of M correlators (corresponding to the M basis functions) followed by the decision maker.**
- **The decision maker employs the minimum distance rule.**

Need M correlators

- **Receiver computes $d_1^2, d_2^2, \ldots, d_M^2$**

Decide s_1 when

$$d_1^2 < d_2^2, d_1^2 < d_3^2, \ldots, d_1^2 < d_M^2$$

Or, equivalently when

$$r_1 > r_2, r_1 > r_3, \ldots, r_1 > r_M$$

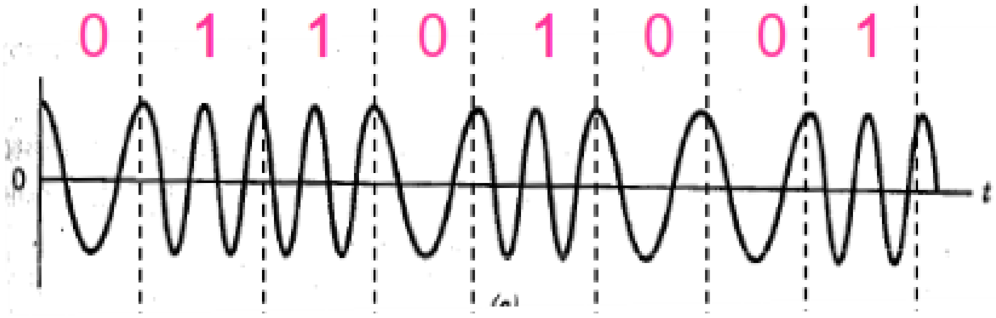
Exercise: Prove the latter equivalency condition

Example 1: Binary FSK

- Modulation

"1" → $s_1(t) = \sqrt{\frac{2E_b}{T_b}} \cos(2\pi f_1 t)$

"0" → $s_2(t) = \sqrt{\frac{2E_b}{T_b}} \cos(2\pi f_2 t)$



$0 \leq t < T_b$

  - E_b : transmitted signal energy per bit

$$\int_0^{T_b} s_1^2(t)dt = \int_0^{T_b} s_2^2(t)dt = E_b$$

  - f_i : transmitted frequency with separation $\Delta f = f_1 - f_0$ $\boldsymbol{\Delta f = \frac{1}{2T_b} = \frac{R_b}{2}}$
  - Δf is selected so that $s_1(t)$ and $s_2(t)$ are orthogonal i.e.

$$\int_0^{T_b} s_1(t)s_2(t)dt = 0$$

Example 1: Binary FSK

Two orthogonal basis functions are required

$$\phi_1(t) = \sqrt{\frac{2}{T_b}} \cos(2\pi f_1 t) \qquad 0 \le t < T_b$$

$$\phi_2(t) = \sqrt{\frac{2}{T_b}} \cos(2\pi f_2 t) \qquad 0 \le t < T_b$$

$$\Rightarrow$$

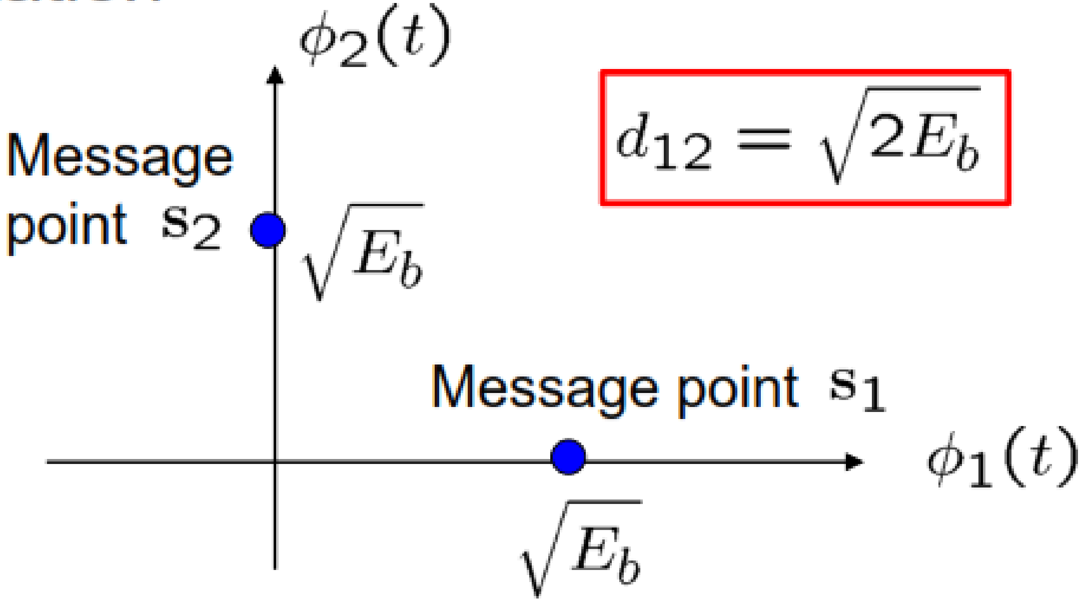
$$s_1(t) = \sqrt{E_b}\,\phi_1(t)$$

$$s_2(t) = \sqrt{E_b}\,\phi_2(t)$$

Signal space representation

$$\mathbf{s}_1 = [\sqrt{E_b} \quad 0]$$

$$\mathbf{s}_2 = [0 \quad \sqrt{E_b}]$$



$$\boxed{d_{12} = \sqrt{2E_b}}$$

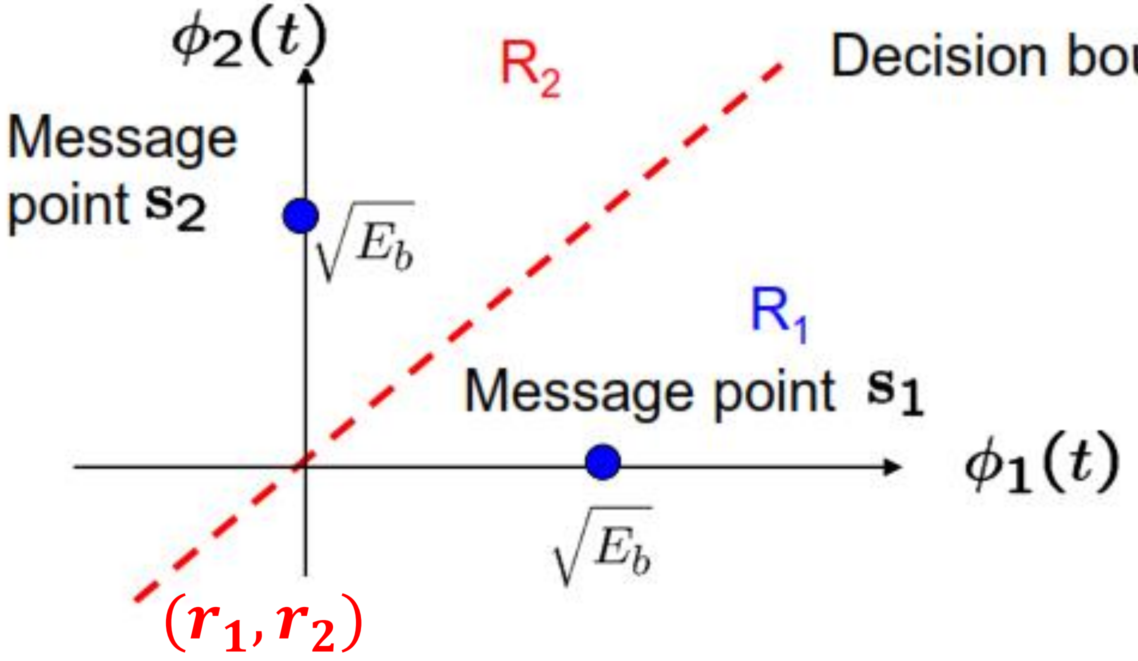
Example 1: Binary FSK

Observation vector

$$\vec{r} = [r_1 \quad r_2]$$

$$r_1 = \int_0^{T_b} r(t)\phi_1(t)dt$$

$$r_2 = \int_0^{T_b} r(t)\phi_2(t)dt$$



Minimum Distance Rule

Decide s1 when

$$\left(r_1 - \sqrt{E}\right)^2 + (r_2)^2 < (r_1)^2 + \left(r_2 - \sqrt{E}\right)^2$$

This rule simplifies to

Decide s1 when

$$r_1 > r_2$$

The receiver decides in favor of s_1 if the observation vector r falls inside region R_1. This occurs when $r_1 > r_2$

When $r_1 < r_2$, r falls inside region R_2 and the receiver decides in favor of s_2

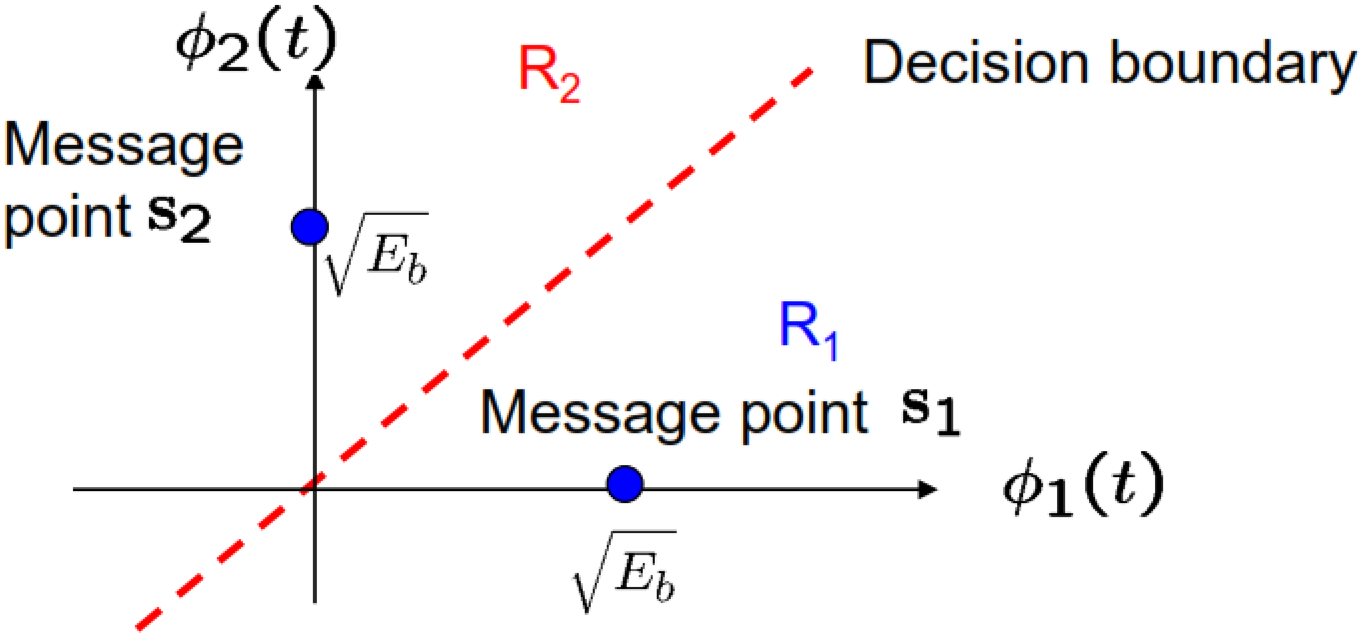
Example 1: Binary FSK

To calculate the error probability, we use the formula:

$$P_s \approx (\text{Number of Signals at distance dmin})\, Q\left(\frac{d_{\min}}{\sqrt{2N_0}}\right) = (1)\cdot Q\left(\frac{\sqrt{2E_b}}{\sqrt{2N_0}}\right) = Q\left(\sqrt{\frac{E_b}{N_0}}\right)$$

where

$$d_{\min} = \sqrt{2E_b}$$



Example 2: 3-ary FSK

To calculate the error probability, we use the formula:

$$P_s \approx (\text{Number of Signals at distance dmin})\, Q\left(\frac{d_{\min}}{\sqrt{2N_0}}\right) = (2)\cdot Q\left(\frac{\sqrt{2E_s}}{\sqrt{2N_0}}\right) = 2Q\left(\sqrt{\frac{E_s}{N_0}}\right)$$

where

$$d_{\min} = \sqrt{2E_s}$$

Minimum Distance Rule

Calculate: $(d_1)^2 = \left(r_1 - \sqrt{E}\right)^2 + (r_2)^2 + (r_3)^2$

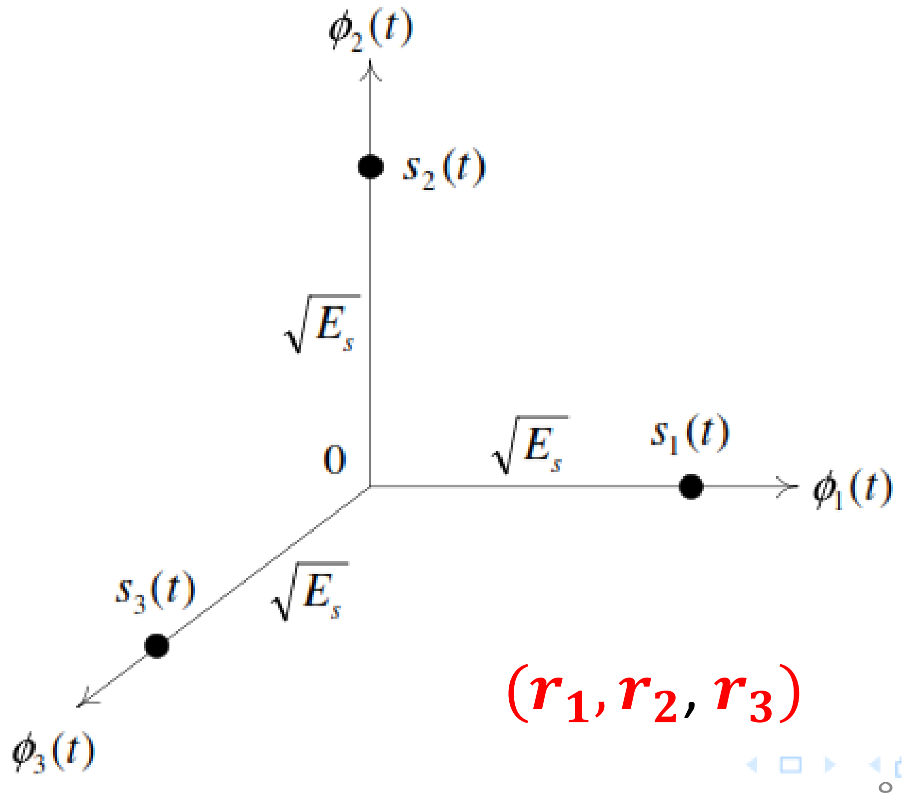
$$(d_2)^2 = (r_1)^2 + \left(r_2 - \sqrt{E}\right)^2 + (r_3)^2$$

$$(d_3)^2 = (r_1)^2 + (r_2)^2 + \left(r_3 - \sqrt{E}\right)^2$$

Choose s1 when $(d_1)^2 < (d_2)^2$ and $(d_1)^2 < (d_3)^2$

Equivalently, Decide s1 when

$r_1 > r_2$ and $r_1 > r_3$



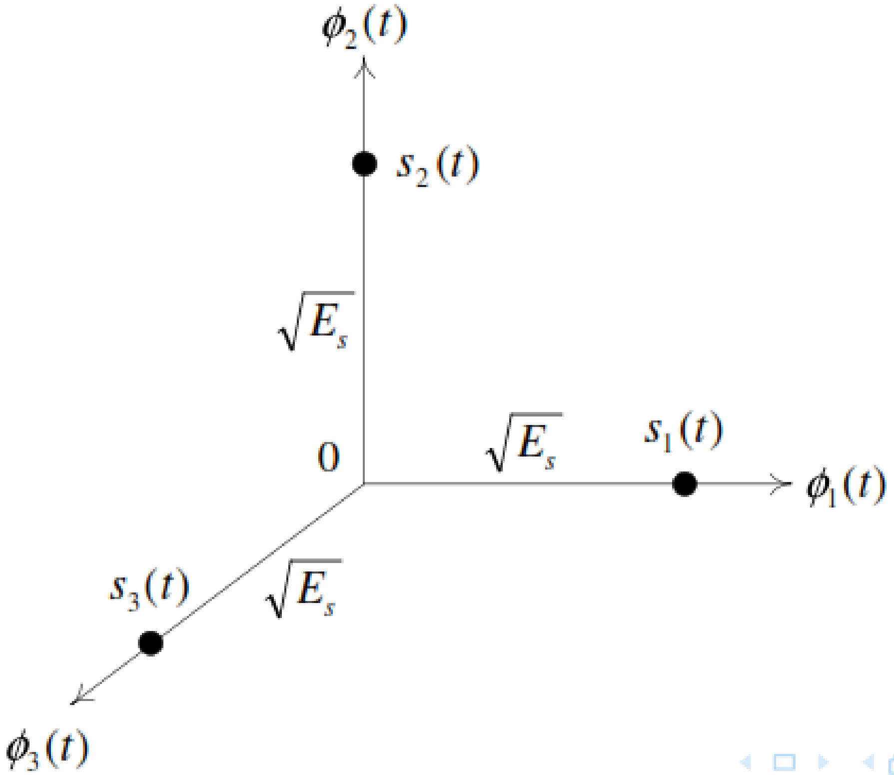
(r_1, r_2, r_3)

Error Probability in an M-ary FSK

To calculate the error probability, we use the formula:

$$P_s \approx (\text{Number of Signals at distance dmin})\ Q\left(\frac{d_{\min}}{\sqrt{2N_0}}\right)$$

$$= (\text{M-1}) \cdot Q\left(\frac{\sqrt{2E_s}}{\sqrt{2N_0}}\right) = (M-1)Q\left(\sqrt{\frac{E_s}{N_0}}\right)$$



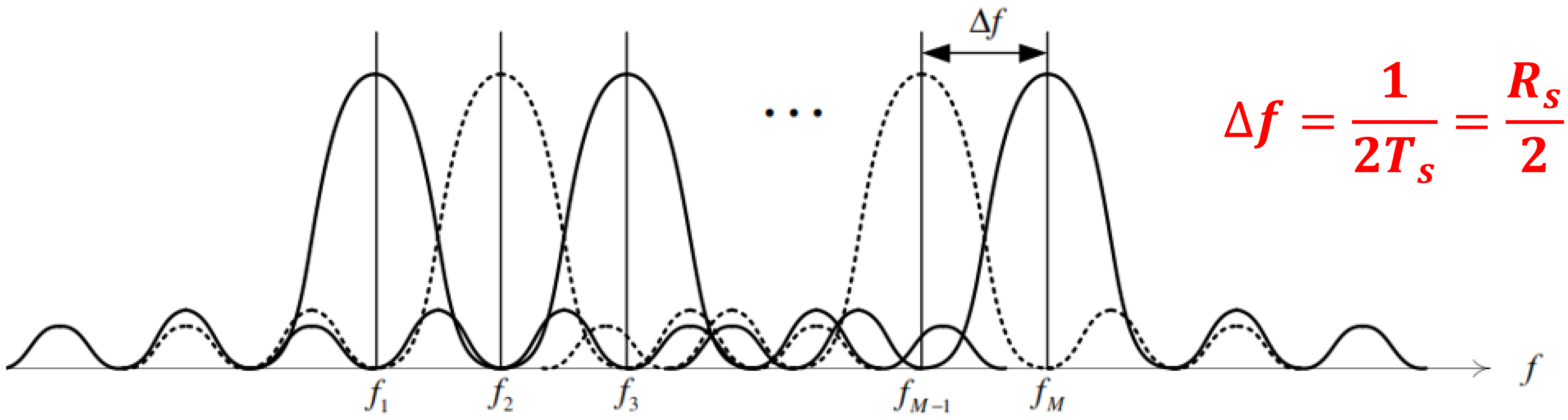
where

$$d_{\min} = \sqrt{2E_s}$$

$$E_s = (\log_2 M)E_b$$

Bandwidth Requirements of M-FSK

- Let $M = 2^{\lambda}$ and let the M signals be orthogonal. The minimum frequency separation between adjacent signals $\Delta f = \frac{R_s}{2}$.
- The bandwidth $B.W = (M-1)\left(\frac{R_s}{2}\right) + 2R_s$.
- For the case when $M = 2, B.W = \left(\frac{R_s}{2}\right) + 2R_s = \frac{5}{2}R_s = \frac{5}{2}R_b$.
- For the case when $M = 4, B.W = \left(\frac{3R_s}{2}\right) + 2R_s = \frac{7}{2}R_s = \frac{7}{2}\frac{R_b}{\log(4)} = \frac{7}{4}R_b$.



$$\Delta f = \frac{1}{2T_s} = \frac{R_s}{2}$$

M-ary Quadrature Amplitude Modulation (M-QAM)

- M-QAM are two-dim constellations and they involve inphase (I) and quadrature (Q) carriers:

$$\phi_I(t) = \sqrt{\frac{2}{T_s}}\cos(2\pi f_c t), \quad 0 \le t \le T_s,$$
$$\phi_Q(t) = \sqrt{\frac{2}{T_s}}\sin(2\pi f_c t), \quad 0 \le t \le T_s,$$

- The ith transmitted M-QAM signal is:

$$s_i(t) = V_{I,i}\sqrt{\frac{2}{T_s}}\cos(2\pi f_c t) + V_{Q,i}\sqrt{\frac{2}{T_s}}\sin(2\pi f_c t), \quad \begin{array}{l} 0 \le t \le T_s \\ i = 1, 2, \ldots, M \end{array}$$
$$= \sqrt{E_i}\sqrt{\frac{2}{T_s}}\cos(2\pi f_c t - \theta_i)$$

$V_{I,i}$ and $V_{Q,i}$ are the information-bearing discrete amplitudes of the two quadrature carriers, $E_i = V_{I,i}^2 + V_{Q,i}^2$ and $\theta_i = \tan^{-1}(V_{Q,i}/V_{I,i})$.

- In general, QAM symbols have different energies. The average symbol energy is calculated as:

$$E_s = \sum_{i=1}^{M} E_i P[s_i(t)] = \frac{\sum_{i=1}^{M} E_i}{M}, \quad \text{for equally-likely signals}$$

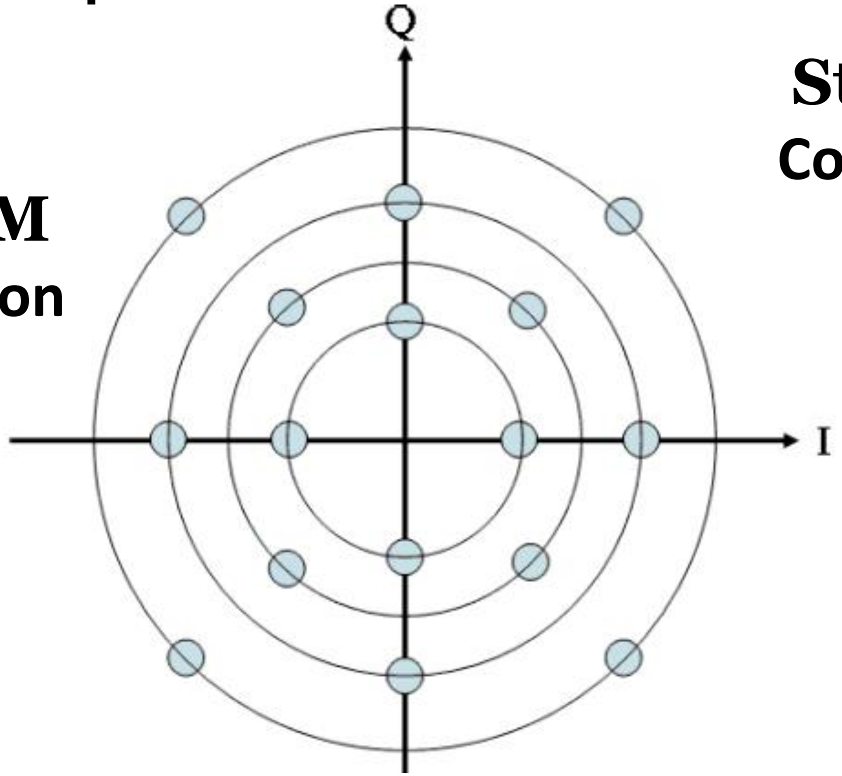
- **In M-QAM, the messages are encoded into both the amplitude and phase of the carrier.**
- **QAM is a two-dimensional encoding scheme and requires two basis functions.**
- **The QAM scheme represents bits as points in a quadrant grid know as a constellation map.**

$$s_i(t) = a_i\phi_1 + b_i\phi_2$$
$E_i = {a_i}^2 + {b_i}^2$ **(prove)**

Criteria for Selecting a Given Constellation

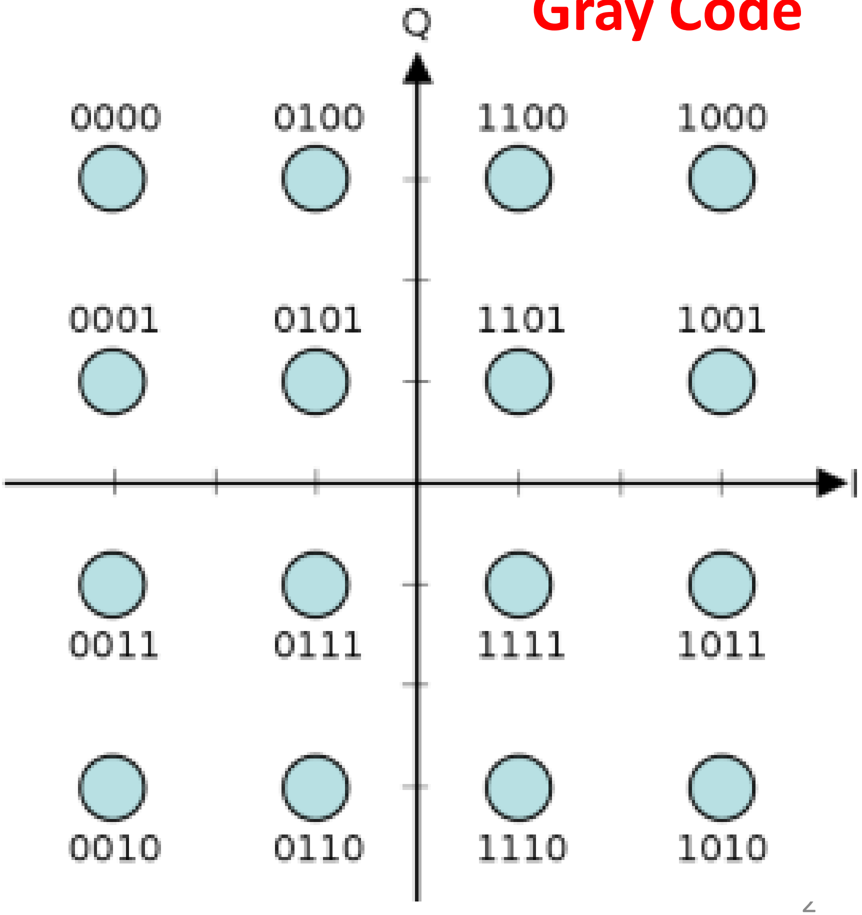
- **Probability of Error**: In signaling over AWGN, the most likely errors are those which confuse a signal with its neighbors. To maintain the same symbol error probability, the **distance between the nearest neighbors are kept the same**.
- **Average Transmitted Energy**: The most efficient signal constellation is the one that has the smallest average transmitted energy.
- **Simplicity in Modulation and Demodulation.**
- **Bandwidth Requirement**.

Star QAM Constellation

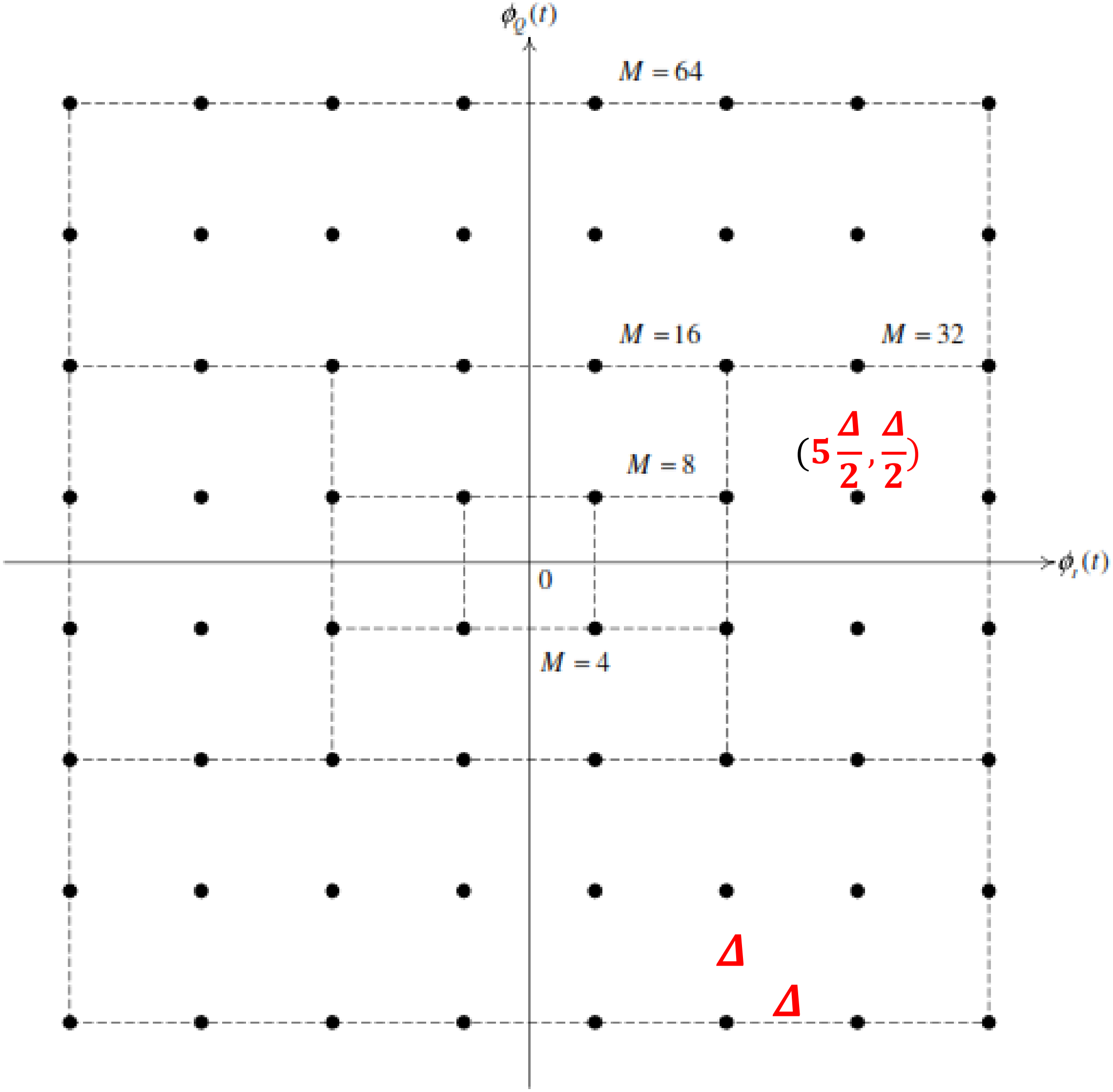


Star QAM Constellation

Gray Code



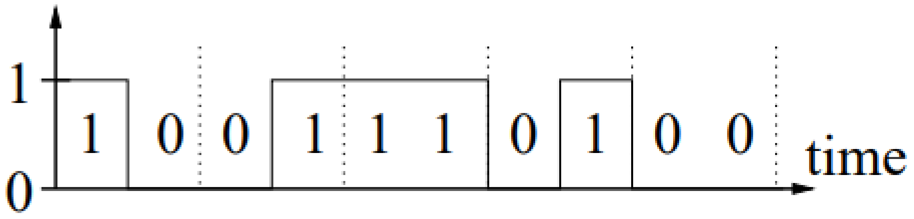
Rectangular M-QAM: Modulation and Demodulation

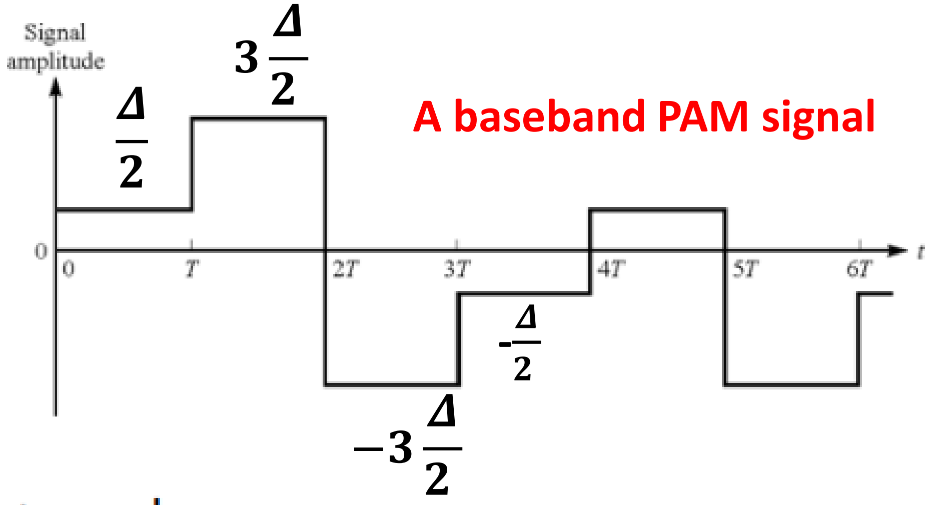


$$s_i(t) = a_i\phi_1 + b_i\phi_2$$

Signal Components $(a_i,\ b_i)$

$$\left\{-7\frac{\Delta}{2}, -5\frac{\Delta}{2}, -3\frac{\Delta}{2}, -\frac{\Delta}{2}, \frac{\Delta}{2}, 3\frac{\Delta}{2}, 5\frac{\Delta}{2}, 7\frac{\Delta}{2}, \right\}$$





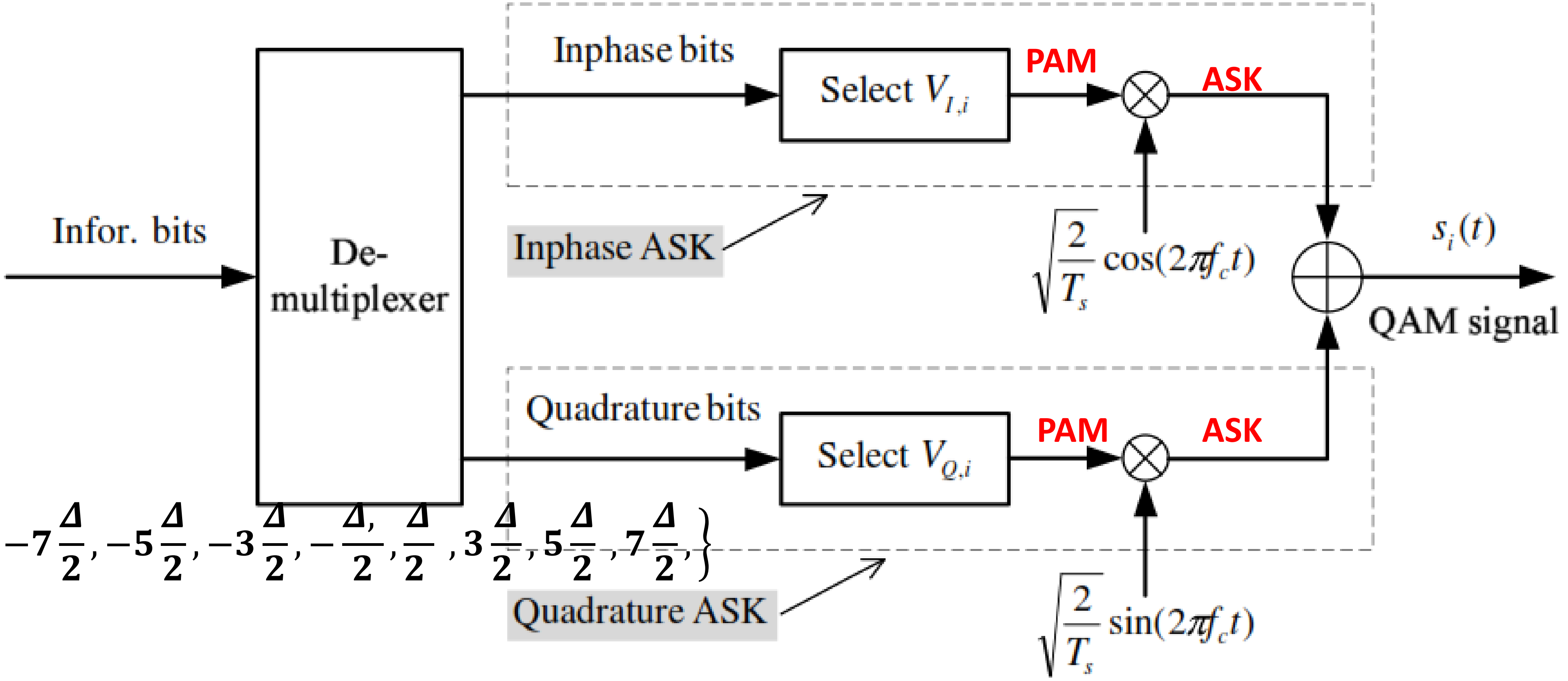
A baseband PAM signal

The signal *components* take value from the set of discrete values $\{(2i-1-M)\Delta/2\}$, $i = 1, 2, \ldots, \frac{M}{2}$.

M-ary QAM Transmitter

- Each group of $\lambda = \log_2 M$ bits can be divided into λ_I inphase bits and λ_Q quadrature bits, where $\lambda_I + \lambda_Q = \lambda$.
- Inphase bits and quadrature bits modulate the inphase and quadrature carriers *independently*.

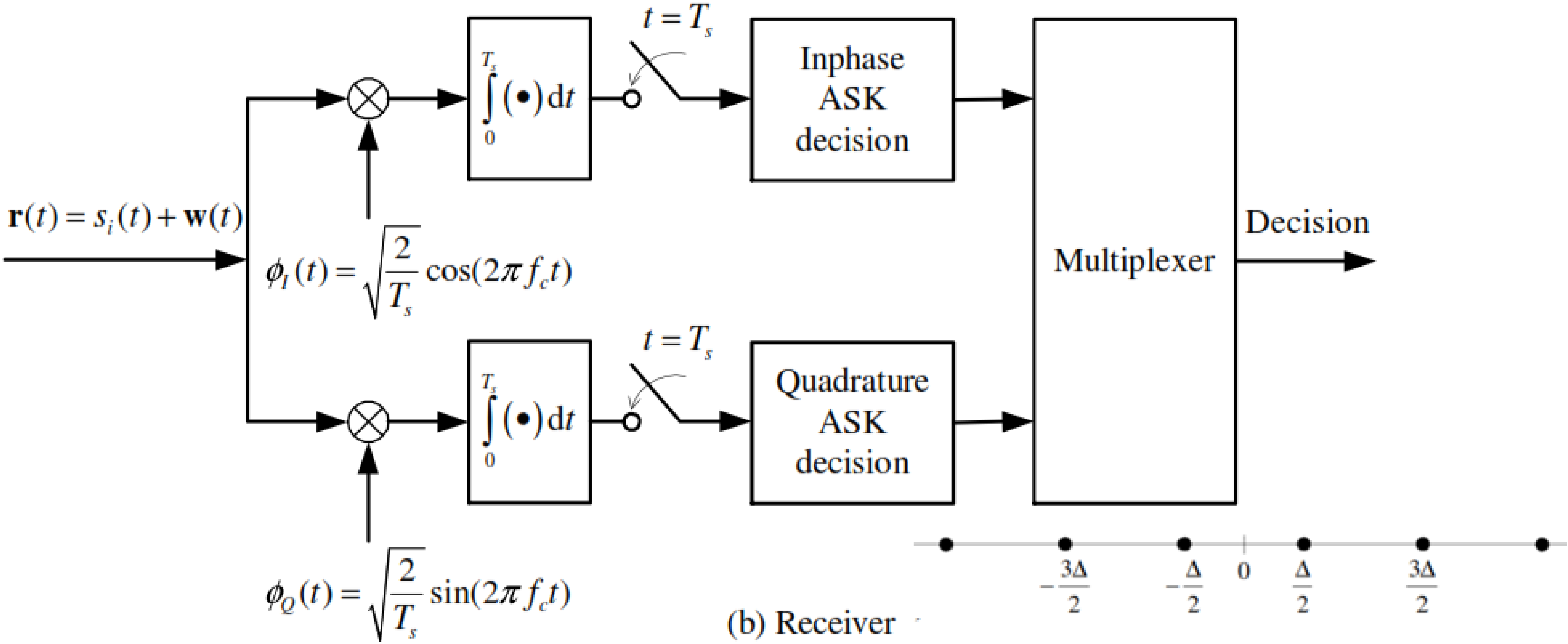
$$\left\{ V_{ij} = -7\frac{\Delta}{2}, -5\frac{\Delta}{2}, -3\frac{\Delta}{2}, -\frac{\Delta}{2}, \frac{\Delta}{2}, 3\frac{\Delta}{2}, 5\frac{\Delta}{2}, 7\frac{\Delta}{2}, \right\}$$



$$\left\{ V_{qj} = -7\frac{\Delta}{2}, -5\frac{\Delta}{2}, -3\frac{\Delta}{2}, -\frac{\Delta}{2}, \frac{\Delta}{2}, 3\frac{\Delta}{2}, 5\frac{\Delta}{2}, 7\frac{\Delta}{2}, \right\}$$

M-ary QAM Receiver

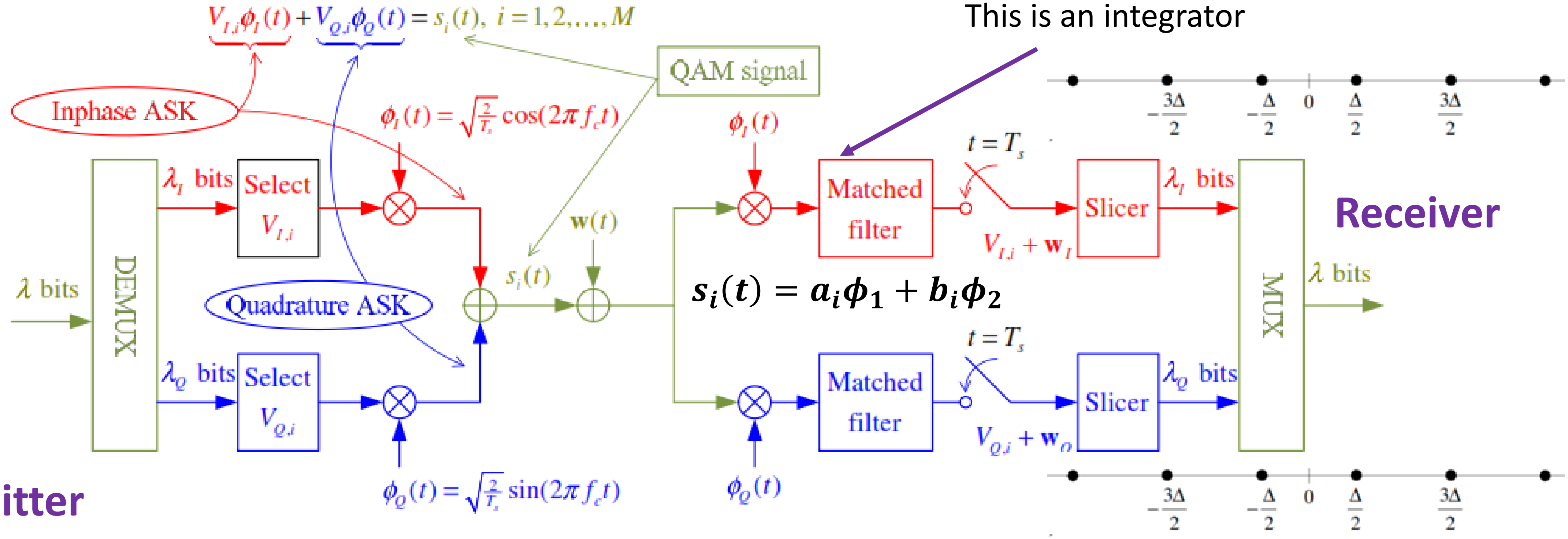
Due to the orthogonality of the inphase and quadrature signals, inphase and quadrature bits can be *independently* detected at the receiver.



(b) Receiver

The most practical rectangular QAM constellation is one which $\lambda_I = \lambda_Q = \lambda/2$, i.e., M is a perfect square and the rectangle is a square.

Implementation of Rectangular M-QAM

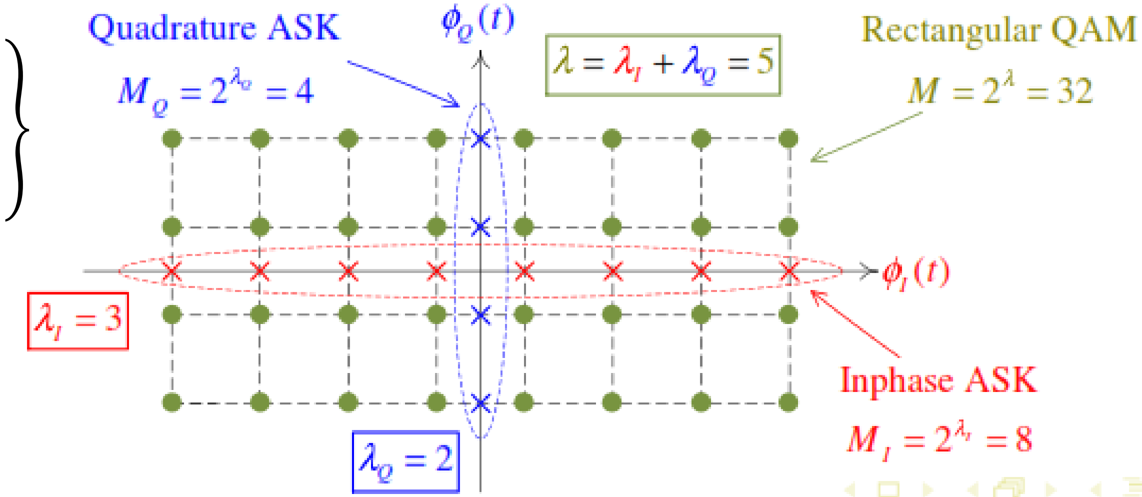


Receiver

Transmitter

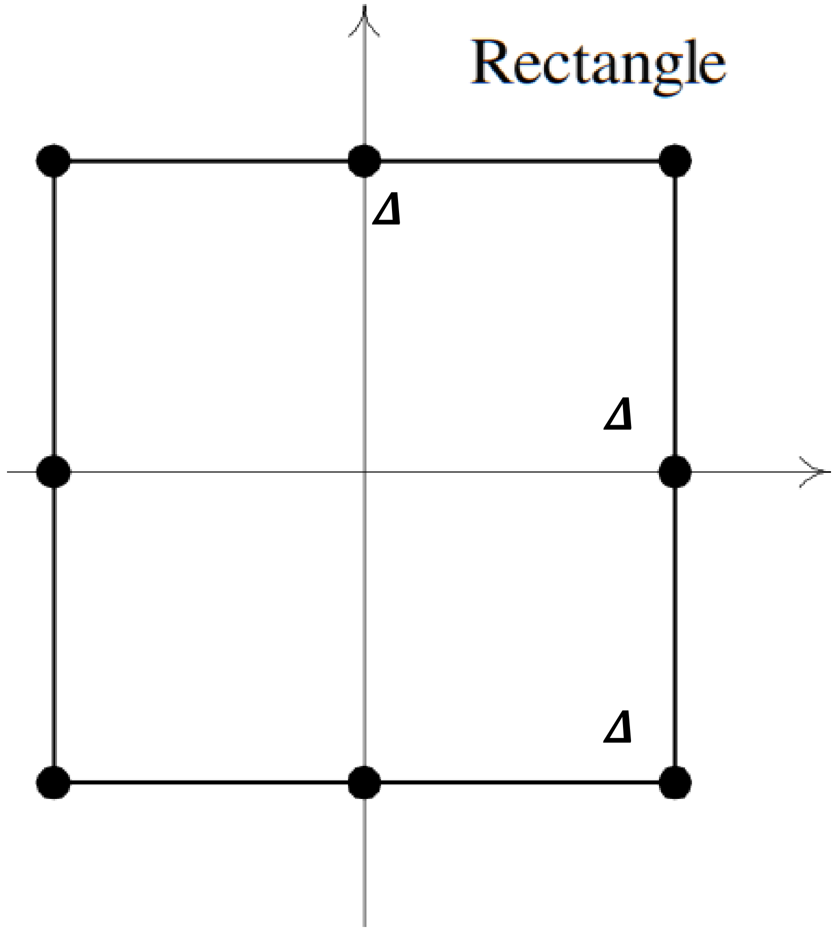
$$V_{ij} = \left\{ -7\frac{\Delta}{2}, -5\frac{\Delta}{2}, -3\frac{\Delta}{2}, -\frac{\Delta}{2}, \frac{\Delta}{2}, 3\frac{\Delta}{2}, 5\frac{\Delta}{2}, 7\frac{\Delta}{2}, \right\}$$

$$V_{qj} = \left\{, -3\frac{\Delta}{2}, -\frac{\Delta}{2}, \frac{\Delta}{2}, 3\frac{\Delta}{2} \right\}$$



Each group of $\lambda = log_2 M$ can be divided into λ_I in-phase bits and λ_Q quadrature bits where $\lambda = \lambda_I + \lambda_Q$.
In-phase and quadrature bits modulate the in-phase and quadrature carriers independently.

M-QAM Constellations: Average Energy and Minimum Distance



$$\boldsymbol{E_{av}} = \frac{\boldsymbol{4\Delta^2+4(2\Delta^2)}}{\boldsymbol{8}} = \boldsymbol{1.5\Delta^2} \Longrightarrow \boldsymbol{\Delta} = \sqrt{\frac{\boldsymbol{E_{av}}}{\boldsymbol{1.5}}}$$

$$\boldsymbol{D_{min}} = \boldsymbol{\Delta}$$

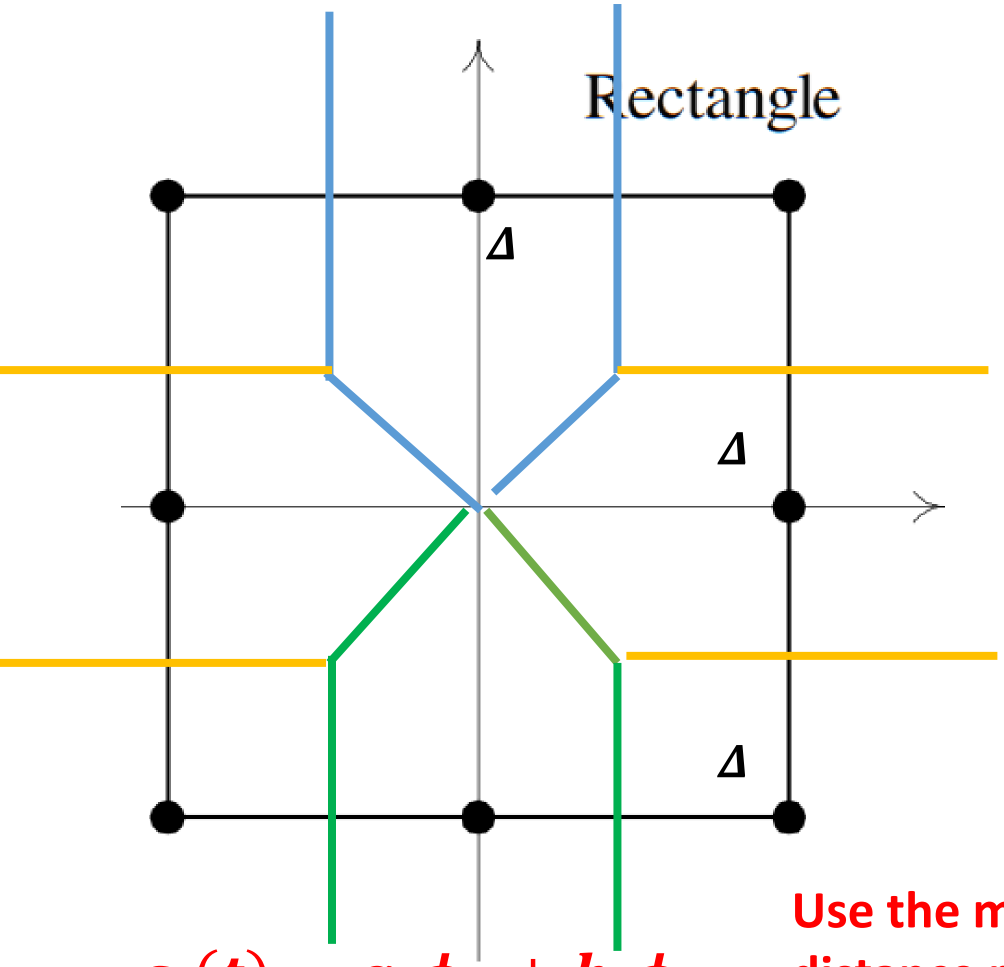
N = Number of neighbors at distance $\boldsymbol{D_{min}}$= 2

$$P_e(m_i) \approx \sum_{\substack{k=1 \\ k\neq i}}^{M} Q\left(\frac{d_{ik}}{\sqrt{2N_0}}\right)$$

$$\mathrm{P}_s \approx (N)\,Q\left(\frac{d_{\mathrm{min}}}{\sqrt{2N_0}}\right) = 2 \cdot Q\left(\frac{d_{\mathrm{min}}}{\sqrt{2N_0}}\right)$$

$$= 2 \cdot Q\left(\frac{\Delta}{\sqrt{2N_0}}\right)$$

M-QAM Constellations: Partitioning of the observation space



$$s_i(t) = a_i\phi_1 + b_i\phi_2$$

$$E_i = {a_i}^2 + {b_i}^2$$

Use the minimum distance rule to partition the observation space among the eight signals.

$$\boldsymbol{E_{av}} = \frac{\mathbf{4\Delta^2 + 4(2\Delta^2)}}{\mathbf{8}} = \mathbf{1.5\Delta^2} \Longrightarrow \boldsymbol{\Delta} = \sqrt{\frac{\boldsymbol{E_{av}}}{\mathbf{1.5}}}$$

$$\boldsymbol{D_{min}} = \boldsymbol{\Delta}$$

N = Number of neighbors at distance $\boldsymbol{D_{min}}$= 2

$$P_e(m_i) \approx \sum_{\substack{k=1 \\ k\neq i}}^{M} Q\left(\frac{d_{ik}}{\sqrt{2N_0}}\right)$$

$$\mathrm{P}_s \approx (N)\, Q\left(\frac{d_{\min}}{\sqrt{2N_0}}\right) = 2 \cdot Q\left(\frac{d_{\min}}{\sqrt{2N_0}}\right)$$

$$= 2 \cdot Q\left(\frac{\Delta}{\sqrt{2N_0}}\right)$$

Symbol Error Probability of M-QAM